

Assignable problems for the Concepts and Practice of Mathematical Finance

Mark Joshi

December 2, 2009

Please note that I have no intention of distributing solutions for these problems. This is to allow those using the book as a text to use them for continuous assessment.

Chapter 2

Question 1. Let S_t be a stock. Show that an American put is worth the same as a European put if interest rates are zero and dividend rate is non-negative.

Question 2. Interest rates are positive, if an American call option on a non-dividend paying stock can be bought for $S - K$, construct an arbitrage.

Chapter 3

Question 3. There are no interest rates. A stock is worth 100 today and will be worth one of the three values 80, 100 and 120 tomorrow. A call option is struck at 90. Give optimal lower and upper bounds on its price. Construct an arbitrage if the option can be sold for your upper bound price.

Question 4. A stock is worth 20 today. Interest rates are zero. The stock can be worth 15, 20 or 25 tomorrow. What are the non-arbitrageable prices for a put option struck at 17?

Question 5. Interest rates are zero. A stock is non-dividend paying. The stock price is currently 40 dollars. Give optimal model-free upper and lower bounds on the price of a portfolio consisting of a digital call struck at 100, a digital call struck at 120, and a digital put struck at 110.

Question 6. Compute

$$\frac{1+x}{1+x+x^2}$$

to $\mathcal{O}(x^3)$ for x small.

Question 7. Interest rates are zero. A stock is worth 100 today and will be worth one of 80, 100 and 120 tomorrow. A put option struck at 100 is worth 5. How much is a call option struck at 110 worth?

Question 8. Compute

$$\frac{x+x^2}{\sqrt{1+x^2}}$$

to $\mathcal{O}(x^4)$ for x small.

Question 9. Compute

$$\frac{1+x+x^2}{\sqrt{1+x^3}}$$

to $\mathcal{O}(x^5)$ for x small.

Question 10. A riskless bond is worth e^{rt} at time t . A stock is worth S_t at time t . We have $0 < t_1 < t_2 < t_3$. Today is time 0. At time t_3 , a derivative, D , pays the average of the value at the times t_1, t_2 and t_3 . What can be said about the price of D today? Justify your answer.

Question 11. The stock price is 100. A riskless bond exists. Zero interest rates. The stock is non-dividend paying. Call options struck at 90, 100, and 110 trade in the market with prices denoted C_{90}, C_{100}, C_{110} . In each of the following cases, construct a static arbitrage or prove that none exists.

C_{90}	C_{100}	C_{110}
0.5	0.11	0.1
10.2	0.2	0.1
10.2	0.2	0.25
11.2	0.2	0.1

Chapter 4

Question 12. Let S_t be the price of a non-dividend paying stock. A derivative, D , pays $|S_T - K|$ at time $T = 0.25$. With the following parameters, give the

approximate price and vega of D ,

$$\begin{aligned}S_0 &= 10, \\K &= 10, \\r &= 0, \\\sigma &= 0.15.\end{aligned}$$

Question 13. A market crash occurs. After the crash option implied vols jump upwards, and the stock price has dropped 30%. For each of the following investors, what can you say about their profit and loss on the day of the crash?

- a person holding a long put position and not hedging;
- a person holding a long put position and delta hedging;
- a person holding a short put position and delta hedging.

Give brief justification.

Question 14. Sketch the gamma of a put option for maturities 0.1, 0.25, 1, and 2 on the same graph. Discuss the qualitative features.

Question 15. In the Black–Scholes model, we have the following parameters

$$\begin{aligned}S &= 10, \\r &= 0, \\\sigma &= 10\%, \\T &= 0.25.\end{aligned}$$

Let P be a put option struck at 10. What is the approximate change in its value for a 1% change in volatility?

Chapter 5

Question 16. A non-dividend-paying stock has price 100. Assume the Black–Scholes model holds with $r = 0, \sigma = 10\%$. A trader sells a put option with one-year expiry, struck at 100 for 4. The Black–Scholes price is 3.98. The trader continuously delta hedges to maturity using a volatility of 10% to compute the delta. In each of the following cases, state how much the trader money has made or lost for the bank after the final pay-off has been paid and hedges dissolved.

- The stock finishes at 90.
- The stock finishes at 100.
- The stock finishes at 110.

Give brief justification.

Question 17. Suppose

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

find the process for X_t^β for some $\beta > 0$. What sort of distribution will X_1^β have?

Question 18. Let S_t be the price of a non-dividend paying stock. A derivative D pays $(K - S_T)_+$ at time T . A trader sells D . She prices and hedges using the Black–Scholes model. A banking crisis hits. The banking regulator then changes the rules so that no further hedging can be carried out. Implied and realized volatilities soar. Discuss the likely immediate effects on the trader’s position. If the crisis continues until time T and no further trading occurs, discuss the final position of the trade. (The answer should be strictly qualitative.) Her friend is in the same situation except that he has traded a contract paying $(S_T - K)_+$, discuss his final position.

Question 19. Two assets X and Y are driven by the same Brownian motion and follow processes as follows

$$dX_t = \sigma X_t dW_t,$$

$$dY_t = \nu Y_t dW_t.$$

Find the process for X_t/Y_t .

Chapter 6

Question 20. A rate, X , is assumed to follow the process

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

in the martingale measure. If A_t is the numeraire, and the tradable bond B_t has value

$$\frac{A_t}{1 + X_t},$$

find the value of $\mu(t, X_t)$ in the martingale measure.

Question 21. A stock, S_t , follows Black–Scholes assumptions. A contract pays $S_T(\log S_T)^2$ at time T , derive a formula for its price.

Question 22. A stock, S_t , follows Black–Scholes assumptions. Let F_t denote the forward price of a contract with expiry T . What is the process followed by F_t in the real-world measure?

Question 23. A perpetual American digital call contract pays 1 at the first time a stock touches the level 10. If the stock is worth 4 today, show that 0.4 is an upper bound for the price of the contract.

Question 24. A stock has time-dependent volatility of $0.1 + 0.1t^2$. What will be the implied volatility of a two-year call option?

Question 25. A hedge fund has been using the Black–Scholes model to price short-dated options. It intends to move into long-dated options. Discuss whether continuing with the Black–Scholes model is wise, and what the alternatives are.

Question 26. Let S_t be a dividend-paying stock with dividend rate q . A contract D pays off if and only if $S_{t_1} > H$. The pay-off is $S_{t_1} - K_1$ at time t_1 and $S_{t_2} - K_2$ at time t_2 . Assuming the Black–Scholes model, develop a formula for the price of D in terms of the cumulative normal function and the bivariate cumulative normal function.

Question 27. Let S_t be the price process of a dividend-paying stock in the Black–Scholes model. Price a contract that pays $S_{t_2}S_{t_1}$ at time t_3 with $0 < t_1 < t_2 < t_3$.

Question 28. Let S_t be a dividend-paying stock in the Black–Scholes model. What are the dynamics of S_t in the martingale measure if we take a delivery contract on S_t as numeraire.

Question 29. Let S_t be a non-dividend paying stock in the Black–Scholes model. Use multiple changes of numeraire to price a contract with pay-off $S_t^l(S_t - K)_+$.

Question 30. Assume the Black–Scholes (BS) model holds. A contract D pays S_T^3 at time T . At time zero:

- find the price of this contract;
- find a formula for the initial hedges (in the BS model) for a trader who is short this contract;
- if the trader is also allowed to vega hedge with an at-the-money call option, find formulas for his hedges.

(Your answers may involve expressions such as N, N', d_1 , but should not further involve any expectations or probabilities.)

Question 31. A stock has the following price process with $X_0 = 0$,

$$X_j = \left(1 - \frac{j}{10}\right)^2 Y,$$

$$Y = \begin{cases} 1 & \text{with } p = 0.75, \\ -1 & \text{with } p = 0.25, \end{cases}$$

for $j = 1, \dots, 10$. A riskless bond which is worth 1 in all states also exists. Either construct an arbitrage or show that none exists in each of the following cases:

- You can trade at times zero and ten only.
- You can trade at any 2 times of your choice.
- You can trade any number of times.

Chapter 7

Question 32. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t + \sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t in order to price on a tree with four branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t} X,$$

where X takes values

$$2u, u, -u, -2u,$$

with probabilities p_1, p_2, p_2 and p_1 respectively. What conditions on p_1, p_2 and u will make prices on this tree converge correctly? How many solutions will there be? If we make the additional assumption that p_1 is equal to p_2 , find the value of u . Will the third and fourth moments of the discretization of the Brownian motion then be correct?

Question 33. A random generator of $N(0, 1)$ variates produces the following draws:

$$0.33, 0.75, 1.42, -0.90, -0.80.$$

What would these draws become after anti-thetic sampling and second moment matching?

(Give each answer to two decimal places.)

Question 34. A Monte Carlo simulation to estimate the price of a complex derivative has standard error of 0.01 after 10,000 paths which takes 100 seconds. If we have 1 hour to find a price, what is the best standard error that can be achieved and how many paths should we run?

Question 35. In a market the observed smile for call options expiring at time T on a non-dividend paying stock near $K = K_0$ is

$$\hat{\sigma}(K) = \alpha + \beta K + \gamma K^2,$$

for some α, β and γ . The discount factor for time T is Z . Find an expression for the price of a digital call option struck at K_0 . (The expression may use the Black–Scholes formula, its Greeks, the cumulative normal function, Z , and S_0 , as well as α, β and γ .)

Question 36. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t + \sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t , to price on a tree with four branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t} X,$$

where X takes values

$$\alpha u, u, -u, -\alpha u,$$

with equal probabilities. What conditions on α and u will make prices on this tree converge correctly? How many solutions will there be? What extra conditions could be imposed to make the solution unique?

Question 37. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t + \sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t to price on a tree with five branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t}X,$$

where X takes values

$$\alpha u, u, 0, -u, -\alpha u,$$

with equal probabilities. What conditions on α and u will make prices on this tree converge correctly? How many solutions will there be? What extra conditions could be imposed to make the solution unique?

Question 38. Let S_t be the price of a non-dividend paying stock. Assume that we can observe the prices of call options, $C(K, T)$, with all strikes, K , and maturities, T . Assume also that we know the prices of zero-coupon bonds of all maturities. Describe how you would find the price of a derivative that pays off 1 dollar for each day that S_t lies between 100 and 110 at closing time during the next year. All cash-flows occur at time 1.

Chapter 8

Question 39. Let S be a non-dividend paying stock. Order the prices of the following contracts as far as possible making only the assumption of no arbitrage.

1. A European call option on S struck at 100 with expiry 1.
2. A European call option on S struck at 101 with expiry 1.
3. A European call option on S struck at 101 with expiry 2.
4. An American call option on S struck at 100 with expiry 1.
5. A down-and-out call option with strike at 101 and barrier at 90 which is continuously monitored. One year expiry.
6. A down-and-out call option with strike at 101 and barrier at 90 which is monitored at times 0.25, 0.5 and 0.75. One year expiry.

Question 40. An asset price follows the process

$$\frac{dS_t}{S_t} = \frac{1}{2}\sigma^2 dt + \sigma dW_t,$$

with $S_0 = 100, T = 1$. Find an expression in terms of the cumulative normal for the probabilities of the following events:

1. $S_T > 100$;
2. $S_T < 81$;
3. $\min_{t \in [0, T]} S_t < 90$;
4. $\min_{t \in [0, T]} S_t < 90, S_T > 100$.

Question 41. A stock follows the process

$$dS_t = \sigma dW_t,$$

with $S_0 = 0$. Find an expression in terms of the cumulative normal for the probability both the following occur:

$$S_T > 0$$

and the minimum of S_t on $[0, T]$ is less than -1 .

Chapter 9

Question 42. A barrier option knocks-out only when the spot has been behind the barrier for 30 consecutive days. Discuss how to price this with Monte Carlo and trees in the Black–Scholes model.

Question 43. A barrier option knocks-out only when the spot has been behind the barrier for a total of 30 days. Discuss how to price this with Monte Carlo and trees in the Black–Scholes model. How will its price compare with the option that requires 30 consecutive days to knock out?

Question 44. Let S_t be the price process of a dividend-paying stock. For $t = t_1, t_2$, with $0 = t_0 < t_1 < t_2$, you can observe today (i.e., time 0) the price of a zero-coupon bond with expiry t , $Z(t)$, the implied volatility, $\hat{\sigma}_t$, of an at-the-money call option expiring at time t , and the forward price f_t for time t . Explain how you would implement a pricing model for the following contract.

- The investor pays 1 to the bank at time zero.
- At time t_1 , if $S_{t_1} < S_{t_0}$ the investor receives 1 and the deal terminates. Otherwise the investor receives a coupon C_1 and the deal continues.

- At time t_2 , the investor receives $1 + S_{t_2}/S_{t_0}$.

The model should return a positive number if the deal is profitable to the bank. Your model should give a price in terms of simple mathematical functions (possibly including the cumulative normal and the bivariate cumulative normal) as functions of the market-observable inputs.

Question 45. A derivative D pays a function f of the stock price at time T , where

$$f(S_T) = \begin{cases} 0 & \text{for } S_T < 100, \\ 1 & \text{for } 100 \leq S_T < 110, \\ S_T - 109 & \text{for } S_T \geq 110. \end{cases}$$

If we can observe the implied volatilities of options of all strikes, and the discount factor for time T , give the price of D in terms of the Black–Scholes formula, its Greeks, cumulative normals and the market-given information.

Question 46. Find upper triangular pseudo-square roots of the following matrices

$$\begin{pmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 6 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

Question 47. Find a pseudo-square root of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix}.$$

Chapter 10

Chapter 11

Question 48. The non-dividend paying stock prices X_t and Y_t follow geometric Brownian motions. A riskless bond growing at rate r exists. The following contracts all pay out at time T . We have $0 < s < T$. Find the price at time zero of contracts with the following pay-offs:

- $X_T Y_T I_{X_T > K}$;
- $X_T I_{Y_s > H}$.

Question 49. The non-dividend paying stock prices X_t and Y_t follow geometric Brownian motions. A riskless bond growing at rate r exists. Let $t_1 < t_2 < t_3$. Find the price at time zero of a contract that pays

$$X_{t_2} Y_{t_2} I_{X_{t_1} > K}$$

at time t_3 .

Question 50. A pair of non-dividend paying stocks, (X, Y) , is modelled by a two-dimensional Black–Scholes model with correlation coefficient ρ . A quant elects to discretize the correlated underlying Brownian motions by four points $(-\alpha, -\alpha)$, $(-\alpha, \alpha)$, $(\alpha, -\alpha)$ and (α, α) , taking probabilities p, q, q and p , respectively. Find values of α, p and q for a time-step of length Δt which will make the model converge correctly as Δt tends to zero.

Question 51. The non-dividend paying stock prices X_t and Y_t follow geometric (jointly normal) Brownian motions with drifts μ_X, μ_Y and volatilities σ_X , and σ_Y . The correlation coefficient is $\rho > 0$. A riskless bond growing at rate r exists. The following contracts all pay out at time T . Using the multi-dimensional Black–Scholes model, find the prices at time zero of contracts with the following pay-offs:

- X_T/Y_T ;
- $(X_T - Y_T)I_{Y_T > H}$.

Question 52. The USD dollar stock X follows geometric Brownian motion with volatility 0.1 and drift 0.1. Its value today is 100 USD. The value of an AUD in USD is 0.8, and follows geometric Brownian motion. Its drift is 0 and its volatility is 0.2. The two Brownian motions are independent. The AUD continuously compounding risk-free rate is 5% and the USD continuously compounding risk-free rate is 3%.

- Give the price of an option to buy X for 125 AUD with one year expiry in terms of the Black–Scholes formula.
- For what strike in AUD would a call and put have the same value?
- For that strike, give an approximate price in USD.