Assignable problems for the Concepts and Practice of Mathematical Finance

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Please note that I have no intention of distributing solutions for these problems. This is to allow those using the book as a text to use them for continuous assessment.

Chapter 2

Question 1. Let S_t be a stock. Show that an American put is worth the same as a European put if interest rates are zero and dividend rate is non-negative.

Question 2. Interest rates are positive, if an American call option on a non-dividend paying stock can be bought for $S - K$, construct an arbitrage.

Question 3. An American option, A pays $f(S_t)$ when exercised and zero otherwise. It expires at a time T. If f has the following properties

- $f(x) \geq 0$ implies $f'(x) > 0$,
- $f(x) \geq 0$ implies $f''(x) > 0$,

show that if S is non-dividend paying and interest rates are positive, then it is never optimal to early exercise A.

Formulate analogues to these conditions in the case where f is not differentiable and give an example where they apply.

Question 4. A contract pays the daily high temperature in degrees C in Paris on the first of July. This is believed to always be between 15 and 35. The contract is currently trading for 22. The discount factor for 1st July is 0.95. Give optimal no arbitrage bounds on another contract that pays the number of degrees above 25 if that number is positive and zero otherwise.

Question 5. A risky zero coupon bond S pays 1 at time 1 if there is no default, and a random value between 0.1 and 1 if default occurs. It is worth 0.4 at time zero. A riskless zero-coupon bond is worth 0.9 at time zero. Give optimal model-free no arbitrage bounds on the following contracts. Justify your answer.

- a digital call on S struck at 0.9,
- a put on S struck at 0.2.

Question 6. A stock, X, is worth 100 today. It pays no dividends. Interest rates are zero and so a zero-coupon bond with 1 year expiry is worth 1. The expected value of the stock in one-year is 120 and it is normal with standard deviation 10. Price a contract in which you must buy the stock for 100 one year from now. A stock, Y, is the same but has expected value 140. Compare the prices of call options on X and Y.

Question 7. Zero interest rates. X is a non-dividend paying stock. A put option on X struck at 100 with expiry T has value 4. What does this tell us about a put option struck at 50?

Question 8. Let S_t denote a non-dividend paying stock. Let B_t be a riskless bond worth e^{rt} at time t with $r > 0$. Let C_t be the price of a European call option struck at K with expiry T and P_t be the price of a European put option with the same strike and expiry. Let F_t be the value of a forward contract with the same strike and expiry. Which of the following relations always hold for $0 \leq t < T$, if there are no arbitrages? You can assume that the probability of $S_T > K$ is strictly between zero and one at all times $t < T$. Fully justify each answer by writing a coherent proof or constructing a counter-example. (You cannot assume a specific model.)

- $C_t > 0$.
- $P_t > 0$.
- $C_t < S_t$.
- $P_t < S_t$.
- $C_t = P_t$.
- $C_t > S_t - K$.

- $P_t < K$.
- $C_t - P_t > 0$, when $S_t > K$.

Question 9. Suppose S_t is a non-dividend paying stock. A bond, B_t, is worth 1 at all times. Let $T > 0$. Assume that for any non-empty interval $I \subset \mathbb{R}_+$, the probability that $S_T \in I$ is positive, and assume that the principle of no arbitrage holds. Let $S_0 < K_1 < K_2$ and let C_j and P_j denote the prices of calls and puts respectively struck at K_j. Which of the following relations always hold at time 0?

- $C_1 < C_2$;
- $P_1 < P_2$;
- $K_1 C_2 > K_2 C_1$;
- $K_1 P_2 > K_2 P_1$;
- $P_1 > C_1$.

Give brief justification.

Chapter 3

Question 10. There are no interest rates. A stock is worth 100 today and will be worth one of the three values 80, 100 and 120 tomorrow. A call option is struck at 90. Give optimal lower and upper bounds on its price. Construct an arbitrage if the option can be sold for your upper bound price.

Question 11. A stock is worth 20 today. Interest rates are zero. The stock can be worth 15, 20 or 25 tomorrow. What are the non-arbitrageable prices for a put option struck at 17?

Question 12. Interest rates are zero. A stock is non-dividend paying. The stock price is currently 40 dollars. Give optimal model-free upper and lower bounds on the price of a portfolio consisting of a digital call struck at 100, a digital call struck at 120, and a digital put struck at 110.

Question 13. Compute

$$\frac{1+x}{1+x+x^2}$$

as a polynomial to $\mathcal{O}(x^3)$ for x small.

Question 14. Interest rates are zero. A stock is worth 100 today and will be worth one of $80, 100$ and 120 tomorrow. A put option struck at 100 is worth 5. How much is a call option struck at 110 worth?

Question 15. Compute

$$\frac{x + x^2}{\sqrt{1 + x^2}}$$

as a polynomial to $\mathcal{O}(x^4)$ for x small.

Question 16. Compute

$$\frac{1 + x + x^2}{\sqrt{1 + x^3}}$$

as a polynomial to $\mathcal{O}(x^5)$ for x small.

Question 17. A riskless bond is worth e^{rt} at time t. A stock is worth S_t at time t. We have $0 < t_1 < t_2 < t_3$. Today is time 0. At time t_3, a derivative, D, pays the average of the value at the times t_1, t_2 and t_3. What can be said about the price of D today? Justify your answer.

Question 18. The stock price is 100. A riskless bond exists. Zero interest rates. The stock is non-dividend paying. Call options struck at $90, 100$, and 110 trade in the market with prices denoted C_{90}, C_{100}, C_{110}. In each of the following cases, construct a static arbitrage or prove that none exists.

C_{90}	C_{100}	C_{110}
0.5	0.11	0.1
10.2	0.2	0.1
10.2	0.2	0.25
11.2	0.2	0.1

.

Question 19. A stock is worth 100 today. It will take one of the four values $170, 190, 210$ and 230 with probabilities $0.2, 0.3, 0.3$ and 0.2, tomorrow. A riskless bond is worth 1 today and 2 tomorrow. A digital put option struck at 220 is known to be worth $\frac{3}{8}$. Give optimal no-arbitrage bounds on the price of a digital call struck at 185. Give brief justification.

Question 20. Compute

$$\frac{(1 + x^2)^{1/3}}{\sqrt{1 + x}}$$

to order x^3. Express your answer as a polynomial plus a term $\mathcal{O}(x^3)$.

Question 21. A market consists of only two assets X and Y. Today they are both worth 1. Tomorrow, there are three possible states of the world A, B, and C, in which their values are as follows

State	X	Y
A	2	3
B	1	1
C	4	3

Give optimal no arbitrage bounds on the price of a contract that pays 1 in state C and zero otherwise.

Question 22. Compute

$$e^{3x}(2+3x^2)^{-1}$$

to order x^3.

Express your answer as a polynomial plus a term $\mathcal{O}(x^3)$.

Question 23. A stock with price S_0 is non-dividend paying and $S_0 = 100$. A trader can borrow and deposit money with zero interest rate. She observes the prices of call options with a one-year maturity on a broker's screen. She sees that the call options struck at 110, 120, 130 have prices $0.9539, 0.6936$ and 0.2766 respectively. She tries to construct a static arbitrage using these options. Construct such an arbitrage or prove that none exists.

Question 24. A stock S has price 109 today. It will pay a dividend of 10 six months from now. It is subject to special trust rules that mean it can be bought and sold today, but it cannot be traded again till one year has passed. The price of a zero-coupon bond with 6-months expiry is 0.9 and with one-year expiry is 0.8. After one year, it is known that the value of the stock will be one of the three values $100, 120$ and 140. A digital put option on S struck at 101 is trading at 0.2. Give optimal no arbitrage bounds on a call option struck at 130. Justify your answer.

Question 25. A non-dividend paying stock is worth 10. It goes up or down by 1 each day. A riskless bond is worth 1 and goes up by 0.05 each day. Price an American put option struck at 10 with a two-day expiry.

Question 26. Let S_t be the price process of a non-dividend paying stock. Let $0 < T_1 < T_2$ and let $K_1 > K_2$. Let C_j be a call option struck at K_j with expiry T_j. There exists zero-coupon bonds $Z_T(t)$ of all maturities, T, which are a non-increasing function of T. Using the principle of no arbitrage, show that the value of C_1 at time zero is less than that of C_2. You should prove any results that you use. You can assume that the probability that S_t takes values in any sub-interval of the positive real numbers is positive.

Question 27. Let X be a non dividend paying stock worth 200 today. Assume that X is worth 220 with probability 0.5 tomorrrow. and that X is worth 190 with probability 0.5 tomorrrow. A riskless bond B is worth 1 at all times. Price a put option struck at 200 which expires tomorrow.

Question 28. Let X be a non dividend paying stock worth 1 today. Suppose that X is worth 1.2 with probability 0.75 tomorrrow, and X is worth 0.8 with probability 0.25 tomorrrow. A riskless bond B is worth 1 at all times. A digital call option, D, pays 1 if $X > 1$ tomorrow and zero otherwise. Find D's replicating portfolio.

Question 29. Let X be a non-dividend paying stock worth 200 today It goes up 15 or down 5 each day for two days. A bond is worth 1 everywhere. What is the no arbitrage price of a call option struck at 210 with 2 day expiry? You sell the option for the price you just computed and hedge it dynamically as specified for multi-step trees? What is your profit/loss if the stock finishes at 190? what about 230?

Question 30. If in the multi-step model $u = 1.1$, $d = 0.9$, $r = 0.05$, $\Delta t = 0.5$, $S_0 = 1$, what is the price of a call option struck at 1 if there are three steps?

Question 31. A stock starts at 100, it goes up or down 10 each day for two days. A bond is worth 0.9, today, 0.95 tomorrow, 1 on the day after. Price an American put option struck at 100.

Question 32. An American put option can be exercised before expiry if the stock price has been above a level H at least once since the contract started. If the initial stock price is below H how would you price this contract on a binomial tree?

Question 33. The CRR binomial tree takes $\mu = 0$ for the tree in Section 3.7.2 We also have $Z(2) = 0.9$, $\sigma = 0.2$, $S_0 = 100$, $K = 100$. Price a two-year expiry American put using the CRR model. Use two steps.

Question 34. The Jarrow–Rudd binomial tree takes $\mu = r - 0.5\sigma^2$ for the tree in Section 3.7.2 It takes the probability, p, of an up-move in the risk-neutral measure to be precisely 0.5 rather than the risk-neutral value. We also have $Z(2) = 0.9$, $\sigma = 0.2$, $S_0 = 100$, $K = 100$. Price a two-year expiry American put using this model. Use two steps. Price also using the risk-neutral value of p.

Question 35. A call option can be early exercised at any time. It knocks out and can no longer be exercised if the stock price falls below 100. Price it using a three-step risk-neutral Jarrow–Rudd tree. Take $T = 1, S_0 = 100, r = 0.05, \sigma = 0.2, K = 95$.

Question 36. An American up and in barrier call option pays $S_t - K$ when exercised but can only be exercised if for some $s \leq t$, $S_s \geq H$. Using the Jarrow–Rudd risk-neutral model, find the price of this option using 3 steps if $S_0 = 100, K = 100, r = 0.05, H = 110, \sigma = 0.15$, and $T = 1$.

Find also the price of a European put option struck at 100 with the same model and expiry.

Question 37. A stock is worth 100 today. A bond is worth 1 at all times. It can go to $70, 90, 110, 130$ tomorrow with equal probabilities. A digital call struck at 120 pays 1 if the stock finishes above 120 and zero otherwise. It is worth 0.3. A digital put struck at 80 pays 1 if the stock finishes below 80 and zero otherwise. It is worth 0.3. What can we say about the price of a call option struck at 100?

Chapter 4

Question 38. Let S_t be the price of a non-dividend paying stock. A derivative, D, pays $|S_T - K|$ at time $T = 0.25$. With the following parameters, give the approximate price and vega of D,

$$\begin{aligned} S_0 &= 10, \\ K &= 10, \\ r &= 0, \\ \sigma &= 0.15. \end{aligned}$$

Question 39. A market crash occurs. After the crash option implied vols jump upwards, and the stock price has dropped 30%. For each of the following investors, what can you say about their profit and loss on the day of the crash?

- a person holding a long put position and not hedging;
- a person holding a long put position and delta hedging;
- a person holding a short put position and delta hedging.

Give brief justification.

Question 40. Sketch the gamma of a put option for maturities $0.1, 0.25, 1$, and 2 on the same graph. Discuss the qualitative features.

Question 41. In the Black–Scholes model, we have the following parameters

$$\begin{aligned} S =& 10, \\ r =& 0, \\ \sigma =& 10\%, \\ T =& 0.25. \end{aligned}$$

Let P be a put option struck at 10. What is the approximate change in its value for a 1% change in volatility?

Question 42. A trader trades options on a corporate stock. She she has sold a call option. It is deeply in-the-money. A takeover bid on the underlying is announced. The stock price and the implied volatility of call options jump upwards. What effect will this have on the trader's position? Give brief justification.

Question 43. A non-dividend paying stock was worth 100. There was a crash during which delta hedging was not possible and the price fell to 80. Implied volatilities of all call options on the stock doubled. For each of the following traders discuss the likely effects on their profit and loss account, and order them as far as possible. The traders are all delta-hedged initially.

1. Short a put option struck at 90.
2. Short a call option struck at 100.
3. Short a put option struck at 110.
4. Short a call option struck at 120.

Question 44. A stock is worth 10 today. Interest rates are zero. A straddle, D, struck at 10 has 3 months expiry. (A straddle is the sum of a call option and a put option with the same strike.)

If estimated volatility is ten percent, estimate the sensitivity of D to a one percent increase in volatility.

Question 45. A non-dividend paying stock has price process S_t. A contract, D, pays $S_T \log S_T$ at time T. Assuming the Black–Scholes model, develop formulas for it price, delta, vega and rho. What would the initial hedge be for a long position in this contract?

Chapter 5

Question 46. A non-dividend-paying stock has price 100. Assume the Black–Scholes model holds with $r = 0, \sigma = 10\%$. A trader sells a put option with one-year expiry, struck at 100 for 4. The Black–Scholes price is 3.98. The trader continuously delta hedges to maturity using a volatility of 10% to compute the delta. In each of the following cases, state how much the trader money has made or lost for the bank after the final pay-off has been paid and hedges dissolved.

- The stock finishes at 90.
- The stock finishes at 100.
- The stock finishes at 110.

Give brief justification.

Question 47. Suppose

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

find the process for X_t^β for some $\beta > 0$. What sort of distribution will X_1^β have?

Question 48. Let S_t be the price of a non-dividend paying stock. A derivative D pays $(K - S_T)_+$ at time T. A trader sells D. She prices and hedges using the Black–Scholes model. A banking crisis hits. The banking regulator then changes the rules so that no further hedging can be carried out. Implied and realized volatilities soar. Discuss the likely immediate effects on the trader's position. If the crisis continues until time T and no further trading occurs, discuss the final position of the trade. (The answer should be strictly qualitative.) Her friend is in the same situation except that he has traded a contract paying $(S_T - K)_+$, discuss his final position.

Question 49. Two assets X and Y are driven by the same Brownian motion and follow processes as follows

$$\begin{aligned} dX_t &= \sigma X_t dW_t, \\ dY_t &= \nu Y_t dW_t. \end{aligned}$$

Find the process for X_t/Y_t.

Question 50. Suppose
$$dY_t = \mu dt + \sigma Y_t dW_t,$$
and
$$G_t = e^{\frac{1}{2}\sigma^2 t - \sigma W_t},$$
where W_t is a standard Brownian motion. Compute the SDE for $Y_t G_t$.

Question 51. Let W_t be a Brownian motion. Suppose $A_0 = B_0 = C_0 = 0$, and

$$\begin{aligned} dA_t &= \alpha dt + a dW_t \\ dB_t &= \beta dt + b dW_t \\ dC_t &= \gamma dt + c dW_t. \end{aligned}$$

What is the expectation of $A_1 B_1 C_1$?

Question 52. A non-dividend paying stock has price process S_t. A discrete random variable, X, takes one of N values, σ_j. Each value is taken with probability $1/N$. The values σ_j are positive and strictly increasing with j. The stock price S_t is known to follow the Black–Scholes model with volatility determined by X. The value of X is known immediately after any options are traded. Before X is known, what can we say about the price of a call option, C, on S_T struck at K? Suppose it is possible to trade contracts D_l that pay 1 at time T if $X = l$ and 0 otherwise. Suppose further that each contract, D_l, costs e^{-rT}/N. What can we now say about the price of C? Justify your answers.

Question 53. Let W_t be a standard Brownian motion. Solve the stochastic differential equation

$$dX_t = \frac{X_t}{1+t} dt + (2 + 2t) dW_t$$

with the initial condition $X_0 = 1$.

Question 54. A process X_t satisfies the stochastic differential equation

$$dX_t = \frac{t^2}{2}(X_t + 2)dt + t(X_t + 2)dW_t$$

and $X_0 = 3$. Find

- $P(X_3 > 3)$,

- $\mathbb{E}(X_3^2)$.

Give brief justification and you may express your answers in term of the cumulative normal function, N, if you wish.

Question 55. If

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

find the SDE for $S_t^{-1/2}$ and solve it. What is $\mathbb{E}(S_T^{-1/2})$?

Question 56. Suppose W_t, Z_t are independent Brownian motions. Do any of the following define a Brownian motion on $[0, 1]$?

- $C_t = (2W_t + Z_t)/\sqrt{5}$.
- $X_t = t^{-1}W_t^3,\ X_0 = 0$.
- $Y_t = W_2 - \frac{1}{2}W_{2-2t}$.

Question 57. Suppose W_t is a standard Brownian motion. Suppose

$$dY_t = t(Y_t + 2)dt + \sigma(Y_t + 2)dW_t.$$

If $Y_0 = 1$, what is $\mathbb{P}(Y_1 > 1)$?

Question 58. Suppose S_t satisfies Black–Scholes model assumptions. Suppose an option pays on the cube of S_t instead of on S_t. What equation will it satisfy?

Chapter 6

Question 59. A rate, X, is assumed to follow the process

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

in the martingale measure. If A_t is the numeraire, and the tradable bond B_t has value

$$\frac{A_t}{1 + X_t},$$

find the value of $\mu(t, X_t)$ in the martingale measure.

Question 60. A stock, S_t, follows Black–Scholes assumptions. A contract pays $S_T(\log S_T)^2$ at time T, derive a formula for its price.

Question 61. A stock, S_t, follows Black–Scholes assumptions. Let F_t denote the forward price of a contract with expiry T. What is the process followed by F_t in the real-world measure?

Question 62. A perpetual American digital call contract pays 1 at the first time a stock touches the level 10. If the stock is worth 4 today, show that 0.4 is an upper bound for the price of the contract.

Question 63. A stock has time-dependent volatility of $0.1 + 0.1t^2$. What will be the implied volatility of a two-year call option?

Question 64. A hedge fund has been using the Black–Scholes model to price short-dated options. It intends to move into long-dated options. Discuss whether continuing with the Black–Scholes model is wise, and what the alternatives are.

Question 65. Let S_t be a dividend-paying stock with dividend rate q. A contract D pays off if and only if $S_{t_1} > H$. The pay-off is $S_{t_1} - K_1$ at time t_1 and $S_{t_2} - K_2$ at time t_2. Assuming the Black–Scholes model, develop a formula for the price of D in terms of the cumulative normal function and the bivariate cumulative normal function.

Question 66. Let S_t be the price process of a dividend-paying stock in the Black–Scholes model. Price a contract that pays $S_{t_2} S_{t_1}$ at time t_3 with $0 < t_1 < t_2 < t_3$.

Question 67. Let S_t be a dividend-paying stock in the Black–Scholes model. What are the dynamics of S_t in the martingale measure if we take a delivery contract on S_t as numeraire.

Question 68. Let S_t be a non-dividend paying stock in the Black–Scholes model. Use multiple changes of numeraire to price a contract with pay-off $S_t^l (S_t - K)_+$.

Question 69. Assume the Black–Scholes (BS) model holds. A contract D pays S_T^3 at time T. At time zero:

- find the price of this contract;
- find a formula for the initial hedges (in the BS model) for a trader who is short this contract;
- if the trader is also allowed to vega hedge with an at-the-money call option, find formulas for his hedges.

(Your answers may involve expressions such as N, N', d_1, but should not further involve any expectations or probabilities.)

Question 70. A stock has the following price process with $X_0 = 0$,

$$X_j = \left(1 - \frac{j}{10}\right)^2 Y,$$

$$Y = \begin{cases} 1 \text{ with } p = 0.75, \\ -1 \text{ with } p = 0.25, \end{cases}$$

for $j = 1, \ldots, 10$. A riskless bond which is worth 1 in all states also exists. Either construct an arbitrage or show that none exists in each of the following cases:

- You can trade at times zero and ten only.
- You can trade at any 2 times of your choice.
- You can trade any number of times.

Question 71. A rate, X, is assumed to follow the process

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

If

$$\frac{1}{1 + X_t},$$

is a martingale, find the value of $\mu(t, X_t)$ as a function of $\sigma(t, X_t)$.

Question 72. In a certain market, there are two assets X and Y with initial values $X_0 = Y_0 = 0$. There is also a riskless bond, B, always worth 1. There are two times 0 and 1. At time 1, in the real-world measure X and Y move independently, and they change value by plus or minus 1 with equal probability. There is also a derivative D that pays 1 if $X_1 = Y_1 = 1$ and zero otherwise. We have that $D_0 = 0.25$. A contract E pays 2 if $X_1 = 1 = -Y_1$ and -2 if $X_1 = -1 = -Y_1$ and zero otherwise.

- Is the market consisting of X, Y, and B complete? Either show that it is or give an example that shows it is not. Find no arbitrage bounds on the price of E at time zero in this case.

- Is the market consisting of X, Y, B and D complete? Either show that it is or give an example that shows it is not. Find no arbitrage bounds on the price of E at time zero in this case. Find all equivalent martingale measures.

Question 73. Let W_t be a standard Brownian motion. Let λ be a real number. For what values of α and β, if any, is

$$\sinh(\lambda W_t)e^{\alpha t^2+\beta t}$$

a martingale? Justify your answer.

Question 74. Let W_t be a Brownian motion. Suppose

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$
$$dY_t = \nu Y_t dt + \theta Y_t dW_t.$$

What is the expectation of X_T/Y_T? Give your answer as a function of

$$X_0, Y_0, \mu, \nu, \sigma, \theta, T.$$

Question 75. Let S_t be the price process of a non-dividend paying stock. A contract, D, pays $S_T(\log S_T)^2$ at time $2T$. Develop a formula for the price of this contract in the Black–Scholes model with continuously compounding interest rate r and volatility σ.

Question 76. Suppose the random variables R_j are independent, identically distributed and take the three values $-1, 0$ and 2 with probability $1/3$. For what values, if any, of the parameters α, β and γ are each of the following processes martingales?

1. $X_0 = 0$, $X_j = X_{j-1} + R_j + \alpha$,
2. $Y_0 = 10$, $Y_j = Y_{j-1} + R_j^2 + \beta$,
3. $Z_j = X_j^2 + \gamma$.

Question 77. A dividend-paying stock S_t is worth 10 today. A contract involves the client paying 1 today and receiving $(1 + S_{0.5}/S_0)(1 + S_1/S_{0.5})$ at time 1. Develop a formula for the value of this contract to the client. Assume the Black–Scholes model with dividend rate d, interest rate r and volatility σ. If a bank sold this contract, what would its initial delta-hedge be if $r = 0.02, d = 0.02$, and $\sigma = 0.1$?

Question 78. Let W_t be a standard Brownian motion. Suppose X_t and Y_t satisfy

$$dX_t = \mu(X_t + 1)dt + \sigma(X_t + 1)dW_t,$$
$$dY_t = adt + bdW_t.$$

Develop formulas for $\mathbb{E}(X_T Y_T)$, $\mathbb{E}(X_T^2)$ and $\mathbb{E}(Y_T^2)$ in terms of X_0, Y_0, a, b, μ and σ.

Question 79.

- Let X_j be i.i.d. standard normals.
- Let $Y_0 = 1,\ Y_j = Y_{j-1} + X_j^3$.
- Let $Z_0 = 2,\ Z_j = Z_{j-1} + Y_{j-1}X_j$.
- Let $W_j = Y_j^2$.

Which of these are martingales with respect to the information generated by $X_1, \ldots, X_j$?

Question 80. Suppose we let $Y_j = \sum_{i=1}^{j} X_j$ with X_j i.i.d. standard normals. Let $Z_j = Y_j^2 - \mu_j$ with $\mu_j \in \mathbb{R}$ to be chosen. Is it possible to choose μ_j in such a way that Z_j is a martingale?

Question 81. Suppose that there are two risky assets X, Y, and no risk-free asset. At time 0, $X_0 = 100, Y_0 = 50$.

On day 1, either $X_1 = 110$, and $Y_1 = 60$ or $X_1 = 90$, and $Y_1 = 40$. Portfolios of X and Y can be formed with any long or short positions in X and Y.

- Construct an arbitrage or prove that none exist.
- Repeat with $Y_0 = 60$.
- Repeat with $Y_0 = 40$.

Question 82. A contract pays $\sqrt{S_T/S_0}$ at time T. A trader has sold his contract. He immediately puts in a place a Delta hedge. Find this initial hedge in the BS model.

Question 83. Let W_t be a standard Brownian motion. Does

$$4W_t^3 + 2W_t^2 - 12tW_t - 2t$$

define a martingale? Justify your answer.

Question 84. Let W_t be a standard Brownian motion. Suppose f_j for $j = 1, 2$, satisfies

$$df_j = \mu_j(f_1, f_2)dt + \sigma_j dW_t.$$

Suppose also that $\tau > 0$ and

$$\frac{1}{1 + f_1\tau} \text{ and } \frac{1}{(1 + f_1\tau)(1 + f_2\tau)}$$

are martingales. Develop formulas for μ_1 and μ_2 in terms of σ_1, σ_2, f_1, f_2, and τ.

Question 85. Let S_t denote the price of a non-dividend paying stock at time t. A trader sells a contract paying $\log(S_T/S_0)$ at time T. He prices and hedges at all times up to the end of the contract as if the Black–Scholes model were true with a volatility of σ_0. Take $r = 0$. In each of the following cases, discuss his likely final position in terms of profit and loss. In each case, the Black–Scholes model holds bar the exceptions given.

- Volatility is a constant $\sigma_1 < \sigma_0$.
- On one day, there is a crash and S_t loses 10% of its value during a period in which rehedging was not possible.
- On one day, there is a soar and S_t increases its value by 20% during a period in which rehedging was not possible.
- A court ruling appears at time $T/2$ allowing the buyer to early exercise the contract at any time before expiry.

Give brief justifications.

Chapter 7

Question 86. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t+\sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t in order to price on a tree with four branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t}X,$$

where X takes values

$$2u, u, -u, -2u,$$

with probabilities p_1, p_2, p_2 and p_1 respectively. What conditions on p_1, p_2 and u will make prices on this tree converge correctly? How many solutions will there be? If we make the additional assumption that p_1 is equal to p_2, find the value of u. Will the third and fourth moments of the discretization of the Brownian motion then be correct?

Question 87. A random generator of $N(0,1)$ variates produces the following draws:

$$0.33, 0.75, 1.42, -0.90, -0.80.$$

What would these draws become after anti-thetic sampling and second moment matching?
(Give each answer to two decimal places.)

Question 88. A Monte Carlo simulation to estimate the price of a complex derivative has standard error of 0.01 after 10,000 paths which takes 100 seconds. If we have 1 hour to find a price, what is the best standard error that can be achieved and how many paths should we run?

Question 89. In a market the observed smile for call options expiring at time T on a non-dividend paying stock near $K = K_0$ is

$$\hat{\sigma}(K) = \alpha + \beta K + \gamma K^2,$$

for some α, β and γ. The discount factor for time T is Z. Find an expresion for the price of a digital call option struck at K_0. (The expression may use the Black–Scholes formula, its Greeks, the cumulative normal function, Z, and S_0, as well as α, β and γ.)

Question 90. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t+\sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t, to price on a tree with four branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t} X,$$

where X takes values

$$\alpha u, u, -u, -\alpha u,$$

with equal probabilities. What conditions on α and u will make prices on this tree converge correctly? How many solutions will there be? What extra conditions could be imposed to make the solution unique?

Question 91. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t+\sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t to price on a tree with five branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t} X,$$

where X takes values

$$\alpha u, u, 0, -u, -\alpha u,$$

with equal probabilities. What conditions on α and u will make prices on this tree converge correctly? How many solutions will there be? What extra conditions could be imposed to make the solution unique?

Question 92. Let S_t be the price of a non-dividend paying stock. Assume that we can observe the prices of call options, $C(K, T)$, with all strikes, K, and maturities, T. Assume also that we know the prices of zero-coupon bonds of all maturities. Describe how you would find the price of a derivative that pays off 1 dollar for each day that S_t lies between 100 and 110 at closing time during the next year. All cash-flows occur at time 1.

Question 93. An option pays $|S_T - 90|$ at time T. Express its price in terms of the prices of call options and zero-coupon bonds. (You are not allowed to use put options!)

Question 94. A contract pays

$$S_T - 90 \text{ for} S_T < 100, S_T - 100 \text{ for} S_T \geq 100.$$

Express its price in terms of the prices of call options and zero-coupon bonds.

Question 95. You have access to the prices of zero-coupon bonds for all maturities, S_0, the value of a stock today, and the prices, $C(K, T)$, of call options for all strikes K and maturities T. The stock is dividend-paying and has price process S_t. At time T, a contract pays

$$|S_T - 100|$$

if this number is less than 10 and zero otherwise. Find the price of the contract in terms of the data given. If insufficient data has been given, then explain what additional data is necessary and then derive the price.

Chapter 8

Question 96. Let S be a non-dividend paying stock. Order the prices of the following contracts as far as possible making only the assumption of no arbitrage.

1. A European call option on S struck at 100 with expiry 1.
2. A European call option on S struck at 101 with expiry 1.
3. A European call option on S struck at 101 with expiry 2.
4. An American call option on S struck at 100 with expiry 1.
5. A down-and-out call option with strike at 101 and barrier at 90 which is continuously monitored. One year expiry.
6. A down-and-out call option with strike at 101 and barrier at 90 which is monitored at times $0.25, 0.5$ and 0.75. One year expiry.

Question 97. An asset price follows the process

$$\frac{dS_t}{S_t} = \frac{1}{2}\sigma^2 dt + \sigma dW_t,$$

with $S_0 = 100, T = 1$. Find an expression in terms of the cumulative normal for the probabilities of the following events:

1. $S_T > 100$;
2. $S_T < 81$;
3. $\min\limits_{t \in [0,T]} S_t < 90$;
4. $\min\limits_{t \in [0,T]} S_t < 90, S_T > 100$.

Question 98. A stock follows the process

$$dS_t = \sigma dW_t,$$

with $S_0 = 0$. Find an expression in terms of the cumulative normal for the probability both the following occur:

$$S_T > 0$$

and the minimum of S_t on $[0, T]$ is less than -1.

Question 99. Let X_t be the exchange rate between the Australian dollar and the British pound. Let S_t be the price of a non-dividend paying Australian stock. For each of X_t and S_t order the prices of the following contracts as far as possible. All strikes are at the forward price.

1. A European put option with one-year maturity.
2. An American call option with one-year maturity.
3. A European put option with two-year maturity.
4. An American put option with two-year maturity.
5. An American call option with one-year maturity that knocks out if the stock falls below 2/3 of its initial value.

Give brief justification.

Question 100. A contract is worth zero today. Its price obeys

$$dX_t = dW_t$$

with W_t a driftless Brownian motion. A riskless bond can be freely traded at all times and is always worth 1. A derivative contract D, pays 2 at time 2 if during $0 < t < 2$, X_t passes below -1, then above 1 and is below 0 at time 2. It pays 1 otherwise. Develop a formula for the price of D.

Chapter 9

Question 101. A barrier option knocks-out only when the spot has been behind the barrier for 30 consecutive days. Discuss how to price this with Monte Carlo and trees in the Black–Scholes model.

Question 102. A barrier option knocks-out only when the spot has been behind the barrier for a total of 30 days. Discuss how to price this with Monte Carlo and trees in the Black–Scholes model. How will its price compare with the option that requires 30 consecutive days to knock out?

Question 103. Let S_t be the price process of a dividend-paying stock. For $t = t_1, t_2$, with $0 = t_0 < t_1 < t_2$, you can observe today (i.e., time 0) the price of a zero-coupon bond with expiry t, $Z(t)$, the implied volatility, $\hat{\sigma}_t$, of an at-the-money call option expiring at time t, and the forward price f_t for time t. Explain how you would implement a pricing model for the following contract.

- The investor pays 1 to the bank at time zero.
- At time t_1, if $S_{t_1} < S_{t_0}$ the investor receives 1 and the deal terminates. Otherwise the investor receives a coupon C_1 and the deal continues.
- At time t_2, the investor receives $1 + S_{t_2}/S_{t_0}$.

The model should return a positive number if the deal is profitable to the bank. Your model should give a price in terms of simple mathematical functions (possibly including the cumulative normal and the bivariate cumulative normal) as functions of the market-observable inputs.

Question 104. A derivative D pays a function f of the stock price at time T, where

$$f(S_T) = \begin{cases} 0 \text{ for } S_T < 100, \\ 1 \text{ for } 100 \leq S_T < 110, \\ S_T - 109 \text{ for } S_T \geq 110. \end{cases}$$

If we can observe the implied volatilities of options of all strikes, and the discount factor for time T, give the price of D in terms of the Black–Scholes formula, its Greeks, cumulative normals and the market-given information.

Question 105. Find upper triangular pseudo-square roots of the following matrices

$$\begin{pmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 6 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

Question 106. Find a pseudo-square root of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix}.$$

Question 107. A stock is non-dividend paying and has price process S_t. The stock price, S_0, is 100. A riskless bond exists with continuously compounding rate 0.0. Call options struck at 100 with expiry 1 and 2 have implied volatilities 12 and 15 percent respectively. Find the price of a contract paying $(S_2/S_1 - 1)_+$ at time 2 in an appropriate model. Give your answer in terms of the cumulative normal function.

Question 108. The Black–Scholes model is assumed. A stock is dividend-paying with positive dividend rate d. Interest rates are positive. A knock-in down-and-in American barrier call option has expiry T, barrier H and strike K. Thus the option can be exercised at any time before T provided the stock has been below H at least once during the life of the contract.

- Discuss how you would construct an efficient numerical method to approximate the Black–Scholes price for this contract.
- How would the price of this contract compare to that of an American call option without the barrier feature?
- Would the pricing be easier if the dividend-rate were zero?

Question 109. It is your first day working at VillageBank. They have decided to trade exotic foreign exchange options. You are asked to develop a Monte Carlo pricer. Your predecessor has left you a random number generator that produces standard normal random numbers and not much else. They have subscribed to Bloomsbury and can get the following data

- the prices of ZCBs for all maturities for both currencies,
- the price of call options for all maturities struck at the forward.

Discuss how you would build and calibrate a model to price the following contracts

- an Asian option with weekly monitoring dates and two years expiry,
- a discrete barrier option with daily monitoring which is a down and out call, expiry is two years.

How would the accuracy vary with the number of paths used? How else could the accuracy be increased?

Chapter 10

Chapter 11

Question 110. The non-dividend paying stock prices X_t and Y_t follow geometric Brownian motions. A riskless bond growing at rate r exists. The following contracts all pay out at time T. We have $0 < s < T$. Find the price at time zero of contracts with the following pay-offs:

- $X_T Y_T I_{X_T > K}$;
- $X_T I_{Y_s > H}$.

Question 111. The non-dividend paying stock prices X_t and Y_t follow geometric Brownian motions. A riskless bond growing at rate r exists. Let $t_1 < t_2 < t_3$. Find the price at time zero of a contract that pays

$$X_{t_2} Y_{t_2} I_{X_{t_1} > K}$$

at time t_3.

Question 112. A pair of non-dividend paying stocks, (X, Y), is modelled by a two-dimensional Black–Scholes model with correlation coefficient ρ. A quant elects to discretize the correlated underlying Brownian motions by four points $(-\alpha, -\alpha)$, $(-\alpha, \alpha)$, $(\alpha, -\alpha)$ and (α, α), taking probabilities p, q, q and p, respectively. Find values of α, p and q for a time-step of length Δt which will make the model converge correctly as Δt tends to zero.

Question 113. The non-dividend paying stock prices X_t and Y_t follow geometric (jointly normal) Brownian motions with drifts μ_X, μ_Y and volatilities σ_X, and σ_Y. The correlation coefficient is $\rho > 0$. A riskless bond growing at rate r exists. The following contracts all pay out at time T. Using the multi-dimensional Black–Scholes model, find the prices at time zero of contracts with the following pay-offs:

- X_T / Y_T;
- $(X_T - Y_T) I_{Y_T > H}$.

Question 114. The USD dollar stock X follows geometric Brownian motion with volatility 0.1 and drift 0.1. Its value today is 100 USD. The value of an AUD in USD is 0.8, and follows geometric Brownian motion. Its drift is 0 and its volatility is 0.2. The two Brownian motions are independent. The AUD continuously compounding risk-free rate is 5% and the USD continuously compounding risk-free rate is 3%.

- Give the price of an option to buy X for 125 AUD with one year expiry in terms of the Black–Scholes formula.

- For what strike in AUD would a call and put have the same value?
- For that strike, give an approximate price in USD.

Question 115. Three non-dividend paying stocks' price processes $S_t^{(j)}$ obey the multi-dimensional Black-Scholes model. A contract D pays

$$\left(\frac{S_T^{(1)} S_T^{(2)}}{S_T^{(3)}} - S_T^{(3)}\right)_+$$

at time T. Find a formula for the rho of the price of D. (The rho is the derivative of the price with respect to the risk-free rate.)

Question 116. Two non-dividend paying stocks X_t and Y_t satisfy Black–Scholes assumptions with volatilities σ_X and σ_Y. However X_t is denominated in USD and Y_t is denominated in AUD. Let F_t denote the value of one US dollar in AUD at time t. Denote the continuously rate in AUD by r and that in USD by d. Take $d = 0$. Suppose F_t follows geometric Brownian motion with volatility σ_F. The Brownian motions are jointly normal with pairwise correlations ρ_{FX}, ρ_{FY} and ρ_{XY}. Develop a formula for an option that pays

$$(X_T - Y_T)_+$$

Australian dollars at time T.

Question 117. Let X_t^j be the price process of non-dividend paying stocks in a multi-dimensional Black–Scholes model. Let A_t denote their arithmetic average at time t and G_t their geometric average. For a strike K, develop a formula for a contract that pays

$$(A_T - K) I_{G(T)>K}$$

at time T. Generalize to the case where the averages have weights w_i.

Question 118. Let X_t^j be the price process of non-dividend paying stocks in a multi-dimensional Black–Scholes model. Let A_t denote their arithmetic average at time t and G_t their geometric average. For a strike K and barrier levels H_1 and H_2, develop a formula for a contract that pays

$$(A_{T_2} - K) I_{G(T_1)>H_1} I_{G(T_2)>H_2}$$

at time T. Generalize to the case where the averages have weights w_i.

Question 119. In a certain market, there are only two traded assets X_t and Y_t. Both have initial value 100. Let Z_1, Z_2 be independent random variables that take value 10 with probability 0.75 and -10 with value 0.25. There are two time periods. We have

$$X_1 = X_0 + Z_1,$$
$$X_2 = X_1,$$
$$Y_1 = Y_0,$$
$$Y_2 = Y_1 + Z_2.$$

The market has no arbitrages. A contract, C, pays

$$X_2 + Y_2 - 200.$$

Price it. Find also the price of a second contract, D, that pays

$$\max(X_2 + Y_2, 200).$$

Question 120. You are an Australian investor with accounts in AUD. Let X_t be the value of one USD in AUD. Let $Y_t = X_t^{-1}$. Let S_t be the USD price process of a non-dividend paying US stock. How would you price the following?

- An option that pays $(X_1 - 1)_+$ AUD at time 1.
- An option that pays $(Y_1 - 1)_+$ USD at time 1.
- An option that pays $(S_1 - 1)_+$ USD at time 1.
- An option that pays $(S_1 - 1)_+$ AUD at time 1.

(Make assumptions as required.)

Chapter 13

Question 121. Let $t_j = 0.5j$ for $j = 0, 1, 2, 3$. Let $\tau = 0.5$. Let $N_j = 4 - j$, for $j = 0, 1, 2$. Let L_j be the LIBOR rate for the period t_j to t_{j+1} at observed at time t_j. Variable notional swap contracts, X_r, pay $N_j L_j \tau$ at time t_{j+1} and receive $N_j K_r \tau$ at the same time for some fixed rate K_r for $j = 0, 1, \ldots, r$.

The values of K_r that make the contracts X_r have zero value are as follows

$$6.00\%,\ 5.50\%, 6.50\%.$$

Find the swap-rate for a constant notional swap for the dates t_j with $j = 0, 1, 2, 3$.

Question 122. Given discount factors $1, 0.98, 0.96, 0.94$, for times $0, 0.5, 1, 1.5$, compute the following:

- The forward rates for the time periods t to $t + 0.5$ for $t = 0, 0.5, 1$.
- The swap rates for the times periods t to 1.5 with six monthly payments for $t = 0, 0.5, 1$.

Question 123. You are given the following times and discount factors.

Time	Discount factor
0	1
0.5	0.95
1	0.9
1.5	0.85

Find the par swap rate for a 1.5 year swap starting immediately with six-monthly payments. Find also the price of a cap on the six-month rate running for 1.5 years starting today struck at 10% if implied volatilities of all forward rates are 20%. You may express your answer in terms of sums of cumulative normal functions if you wish.

Chapter 15

Question 124. Let W_t be a standard Brownian motion. Let $X_t = 2W_t + t + 1$. What is the dyadic quadratic variation of X_t on the interval $[0, T]$?