

Assignable problems for the Concepts and Practice of Mathematical Finance

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Please note that I have no intention of distributing solutions for these problems. This is to allow those using the book as a text to use them for continuous assessment.

Chapter 2

Question 1. Let S_t be a stock. Show that an American put is worth the same as a European put if interest rates are zero and dividend rate is non-negative.

Question 2. Interest rates are positive, if an American call option on a non-dividend paying stock can be bought for $S - K$, construct an arbitrage.

Question 3. An American option, A pays $f(S_t)$ when exercised and zero otherwise. It expires at a time T . If f has the following properties

- $f(x) \geq 0$ implies $f'(x) > 0$,
- $f(x) \geq 0$ implies $f''(x) > 0$,

show that if S is non-dividend paying and interest rates are positive, then it is never optimal to early exercise A .

Formulate analogues to these conditions in the case where f is not differentiable and give an example where they apply.

Question 4. A contract pays the daily high temperature in degrees C in Paris on the first of July. This is believed to always be between 15 and 35. The contract is currently trading for 22. The discount factor for 1st July is 0.95. Give optimal no arbitrage bounds on another contract that pays the number of degrees above 25 if that number is positive and zero otherwise.

Question 5. A risky zero coupon bond S pays 1 at time 1 if there is no default, and a random value between 0.1 and 1 if default occurs. It is worth 0.4 at time zero. A riskless zero-coupon bond is worth 0.9 at time zero. Give optimal model-free no arbitrage bounds on the following contracts. Justify your answer.

- a digital call on S struck at 0.9,
- a put on S struck at 0.2.

Chapter 3

Question 6. There are no interest rates. A stock is worth 100 today and will be worth one of the three values 80, 100 and 120 tomorrow. A call option is struck at 90. Give optimal lower and upper bounds on its price. Construct an arbitrage if the option can be sold for your upper bound price.

Question 7. A stock is worth 20 today. Interest rates are zero. The stock can be worth 15, 20 or 25 tomorrow. What are the non-arbitrageable prices for a put option struck at 17?

Question 8. Interest rates are zero. A stock is non-dividend paying. The stock price is currently 40 dollars. Give optimal model-free upper and lower bounds on the price of a portfolio consisting of a digital call struck at 100, a digital call struck at 120, and a digital put struck at 110.

Question 9. Compute

$$\frac{1+x}{1+x+x^2}$$

to $\mathcal{O}(x^3)$ for x small.

Question 10. Interest rates are zero. A stock is worth 100 today and will be worth one of 80, 100 and 120 tomorrow. A put option struck at 100 is worth 5. How much is a call option struck at 110 worth?

Question 11. Compute

$$\frac{x+x^2}{\sqrt{1+x^2}}$$

to $\mathcal{O}(x^4)$ for x small.

Question 12. Compute

$$\frac{1 + x + x^2}{\sqrt{1 + x^3}}$$

to $\mathcal{O}(x^5)$ for x small.

Question 13. A riskless bond is worth e^{rt} at time t . A stock is worth S_t at time t . We have $0 < t_1 < t_2 < t_3$. Today is time 0. At time t_3 , a derivative, D , pays the average of the value at the times t_1, t_2 and t_3 . What can be said about the price of D today? Justify your answer.

Question 14. The stock price is 100. A riskless bond exists. Zero interest rates. The stock is non-dividend paying. Call options struck at 90, 100, and 110 trade in the market with prices denoted C_{90}, C_{100}, C_{110} . In each of the following cases, construct a static arbitrage or prove that none exists.

C_{90}	C_{100}	C_{110}
0.5	0.11	0.1
10.2	0.2	0.1
10.2	0.2	0.25
11.2	0.2	0.1

Question 15. A stock is worth 100 today. It will take one of the four values 170, 190, 210 and 230 with probabilities 0.2, 0.3, 0.3 and 0.2, tomorrow. A riskless bond is worth 1 today and 2 tomorrow. A digital put option struck at 220 is known to be worth $\frac{3}{8}$. Give optimal no-arbitrage bounds on the price of a digital call struck at 185. Give brief justification.

Question 16. Compute

$$\frac{(1 + x^2)^{1/3}}{\sqrt{1 + x}}$$

to order x^3 . Express your answer as a polynomial plus a term $\mathcal{O}(x^3)$.

Question 17. A market consists of only two assets X and Y . Today they are both worth 1. Tomorrow, there are three possible states of the world A, B , and C , in which their values are as follows

State	X	Y
A	2	3
B	1	1
C	4	3

Give optimal no arbitrage bounds on the price of a contract that pays 1 in state C and zero otherwise.

Question 18. Compute

$$e^{3x}(2 + 3x^2)^{-1}$$

to order x^3 .

Express your answer as a polynomial plus a term $\mathcal{O}(x^3)$.

Question 19. A stock with price S_0 is non-dividend paying and $S_0 = 100$. A trader can borrow and deposit money with zero interest rate. She observes the prices of call options with a one-year maturity on a broker's screen. She sees that the call options struck at 110, 120, 130 have prices 0.9539, 0.6936 and 0.2766 respectively. She tries to construct a static arbitrage using these options. Construct such an arbitrage or prove that none exists.

Question 20. A stock S has price 109 today. It will pay a dividend of 10 six months from now. It is subject to special trust rules that mean it can be bought and sold today, but it cannot be traded again till one year has passed. The price of a zero-coupon bond with 6-months expiry is 0.9 and with one-year expiry is 0.8. After one year, it is known that the value of the stock will be one of the three values 100, 120 and 140. A digital put option on S struck at 101 is trading at 0.2. Give optimal no arbitrage bounds on a call option struck at 130. Justify your answer.

Chapter 4

Question 21. Let S_t be the price of a non-dividend paying stock. A derivative, D , pays $|S_T - K|$ at time $T = 0.25$. With the following parameters, give the approximate price and vega of D ,

$$\begin{aligned}S_0 &= 10, \\K &= 10, \\r &= 0, \\\sigma &= 0.15.\end{aligned}$$

Question 22. A market crash occurs. After the crash option implied vols jump upwards, and the stock price has dropped 30%. For each of the following investors, what can you say about their profit and loss on the day of the crash?

- a person holding a long put position and not hedging;

- a person holding a long put position and delta hedging;
- a person holding a short put position and delta hedging.

Give brief justification.

Question 23. Sketch the gamma of a put option for maturities 0.1, 0.25, 1, and 2 on the same graph. Discuss the qualitative features.

Question 24. In the Black–Scholes model, we have the following parameters

$$\begin{aligned} S &= 10, \\ r &= 0, \\ \sigma &= 10\%, \\ T &= 0.25. \end{aligned}$$

Let P be a put option struck at 10. What is the approximate change in its value for a 1% change in volatility?

Question 25. A trader trades options on a corporate stock. She she has sold a call option. It is deeply in-the-money. A takeover bid on the underlying is announced. The stock price and the implied volatility of call options jump upwards. What effect will this have on the trader’s position? Give brief justification.

Question 26. A non-dividend paying stock was worth 100. There was a crash during which delta hedging was not possible and the price fell to 80. Implied volatilities of all call options on the stock doubled. For each of the following traders discuss the likely effects on their profit and loss account, and order them as far as possible. The traders are all delta-hedged initially.

1. Short a put option struck at 90.
2. Short a call option struck at 100.
3. Short a put option struck at 110.
4. Short a call option struck at 120.

Question 27. A stock is worth 10 today. Interest rates are zero. A straddle, D , struck at 10 has 3 months expiry. (A straddle is the sum of a call option and a put option with the same strike.)

If estimated volatility is ten percent, estimate the sensitivity of D to a one percent increase in volatility.

Question 28. A non-dividend paying stock has price process S_t . A contract, D , pays $S_T \log S_T$ at time T . Assuming the Black–Scholes model, develop formulas for its price, delta, vega and rho. What would the initial hedge be for a long position in this contract?

Chapter 5

Question 29. A non-dividend-paying stock has price 100. Assume the Black–Scholes model holds with $r = 0$, $\sigma = 10\%$. A trader sells a put option with one-year expiry, struck at 100 for 4. The Black–Scholes price is 3.98. The trader continuously delta hedges to maturity using a volatility of 10% to compute the delta. In each of the following cases, state how much the trader money has made or lost for the bank after the final pay-off has been paid and hedges dissolved.

- The stock finishes at 90.
- The stock finishes at 100.
- The stock finishes at 110.

Give brief justification.

Question 30. Suppose

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

find the process for X_t^β for some $\beta > 0$. What sort of distribution will X_1^β have?

Question 31. Let S_t be the price of a non-dividend paying stock. A derivative D pays $(K - S_T)_+$ at time T . A trader sells D . She prices and hedges using the Black–Scholes model. A banking crisis hits. The banking regulator then changes the rules so that no further hedging can be carried out. Implied and realized volatilities soar. Discuss the likely immediate effects on the trader’s position. If the crisis continues until time T and no further trading occurs, discuss the final position of the trade. (The answer should be strictly qualitative.) Her friend is in the same situation except that he has traded a contract paying $(S_T - K)_+$, discuss his final position.

Question 32. Two assets X and Y are driven by the same Brownian motion and follow processes as follows

$$\begin{aligned} dX_t &= \sigma X_t dW_t, \\ dY_t &= \nu Y_t dW_t. \end{aligned}$$

Find the process for X_t/Y_t .

Question 33. Suppose

$$dY_t = \mu dt + \sigma Y_t dW_t,$$

and

$$G_t = e^{\frac{1}{2}\sigma^2 t - \sigma W_t},$$

where W_t is a standard Brownian motion. Compute the SDE for $Y_t G_t$.

Question 34. Let W_t be a Brownian motion. Suppose $A_0 = B_0 = C_0 = 0$, and

$$dA_t = \alpha dt + a dW_t$$

$$dB_t = \beta dt + b dW_t$$

$$dC_t = \gamma dt + c dW_t.$$

What is the expectation of $A_1 B_1 C_1$?

Question 35. A non-dividend paying stock has price process S_t . A discrete random variable, X , takes one of N values, σ_j . Each value is taken with probability $1/N$. The values σ_j are positive and strictly increasing with j . The stock price S_t is known to follow the Black–Scholes model with volatility determined by X . The value of X is known immediately after any options are traded. Before X is known, what can we say about the price of a call option, C , on S_T struck at K ? Suppose it is possible to trade contracts D_l that pay 1 at time T if $X = l$ and 0 otherwise. Suppose further that each contract, D_l , costs e^{-rT}/N . What can we now say about the price of C ? Justify your answers.

Chapter 6

Question 36. A rate, X , is assumed to follow the process

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

in the martingale measure. If A_t is the numeraire, and the tradable bond B_t has value

$$\frac{A_t}{1 + X_t},$$

find the value of $\mu(t, X_t)$ in the martingale measure.

Question 37. A stock, S_t , follows Black–Scholes assumptions. A contract pays $S_T(\log S_T)^2$ at time T , derive a formula for its price.

Question 38. A stock, S_t , follows Black–Scholes assumptions. Let F_t denote the forward price of a contract with expiry T . What is the process followed by F_t in the real-world measure?

Question 39. A perpetual American digital call contract pays 1 at the first time a stock touches the level 10. If the stock is worth 4 today, show that 0.4 is an upper bound for the price of the contract.

Question 40. A stock has time-dependent volatility of $0.1 + 0.1t^2$. What will be the implied volatility of a two-year call option?

Question 41. A hedge fund has been using the Black–Scholes model to price short-dated options. It intends to move into long-dated options. Discuss whether continuing with the Black–Scholes model is wise, and what the alternatives are.

Question 42. Let S_t be a dividend-paying stock with dividend rate q . A contract D pays off if and only if $S_{t_1} > H$. The pay-off is $S_{t_1} - K_1$ at time t_1 and $S_{t_2} - K_2$ at time t_2 . Assuming the Black–Scholes model, develop a formula for the price of D in terms of the cumulative normal function and the bivariate cumulative normal function.

Question 43. Let S_t be the price process of a dividend-paying stock in the Black–Scholes model. Price a contract that pays $S_{t_2}S_{t_1}$ at time t_3 with $0 < t_1 < t_2 < t_3$.

Question 44. Let S_t be a dividend-paying stock in the Black–Scholes model. What are the dynamics of S_t in the martingale measure if we take a delivery contract on S_t as numeraire.

Question 45. Let S_t be a non-dividend paying stock in the Black–Scholes model. Use multiple changes of numeraire to price a contract with pay-off $S_t^l(S_t - K)_+$.

Question 46. Assume the Black–Scholes (BS) model holds. A contract D pays S_T^3 at time T . At time zero:

- find the price of this contract;
- find a formula for the initial hedges (in the BS model) for a trader who is short this contract;
- if the trader is also allowed to vega hedge with an at-the-money call option, find formulas for his hedges.

(Your answers may involve expressions such as N , N' , d_1 , but should not further involve any expectations or probabilities.)

Question 47. A stock has the following price process with $X_0 = 0$,

$$X_j = \left(1 - \frac{j}{10}\right)^2 Y,$$

$$Y = \begin{cases} 1 & \text{with } p = 0.75, \\ -1 & \text{with } p = 0.25, \end{cases}$$

for $j = 1, \dots, 10$. A riskless bond which is worth 1 in all states also exists. Either construct an arbitrage or show that none exists in each of the following cases:

- You can trade at times zero and ten only.
- You can trade at any 2 times of your choice.
- You can trade any number of times.

Question 48. A rate, X , is assumed to follow the process

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

If

$$\frac{1}{1 + X_t},$$

is a martingale, find the value of $\mu(t, X_t)$ as a function of $\sigma(t, X_t)$.

Chapter 7

Question 49. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t + \sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t in order to price on a tree with four branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t}X,$$

where X takes values

$$2u, u, -u, -2u,$$

with probabilities p_1, p_2, p_2 and p_1 respectively. What conditions on p_1, p_2 and u will make prices on this tree converge correctly? How many solutions will there be? If we make the additional assumption that p_1 is equal to p_2 , find the value of u . Will the third and fourth moments of the discretization of the Brownian motion then be correct?

Question 50. A random generator of $N(0, 1)$ variates produces the following draws:

$$0.33, 0.75, 1.42, -0.90, -0.80.$$

What would these draws become after anti-thetic sampling and second moment matching?

(Give each answer to two decimal places.)

Question 51. A Monte Carlo simulation to estimate the price of a complex derivative has standard error of 0.01 after 10,000 paths which takes 100 seconds. If we have 1 hour to find a price, what is the best standard error that can be achieved and how many paths should we run?

Question 52. In a market the observed smile for call options expiring at time T on a non-dividend paying stock near $K = K_0$ is

$$\hat{\sigma}(K) = \alpha + \beta K + \gamma K^2,$$

for some α, β and γ . The discount factor for time T is Z . Find an expression for the price of a digital call option struck at K_0 . (The expression may use the Black–Scholes formula, its Greeks, the cumulative normal function, Z , and S_0 , as well as α, β and γ .)

Question 53. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t + \sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t , to price on a tree with four branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t} X,$$

where X takes values

$$\alpha u, u, -u, -\alpha u,$$

with equal probabilities. What conditions on α and u will make prices on this tree converge correctly? How many solutions will there be? What extra conditions could be imposed to make the solution unique?

Question 54. A stock is modelled in the risk-neutral measure of the Black–Scholes model via

$$S_t = S_0 e^{(r-0.5\sigma^2)t + \sigma W_t},$$

with W_t a Brownian motion. A quant discretizes W_t to price on a tree with five branches. We have

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t}X,$$

where X takes values

$$\alpha u, u, 0, -u, -\alpha u,$$

with equal probabilities. What conditions on α and u will make prices on this tree converge correctly? How many solutions will there be? What extra conditions could be imposed to make the solution unique?

Question 55. Let S_t be the price of a non-dividend paying stock. Assume that we can observe the prices of call options, $C(K, T)$, with all strikes, K , and maturities, T . Assume also that we know the prices of zero-coupon bonds of all maturities. Describe how you would find the price of a derivative that pays off 1 dollar for each day that S_t lies between 100 and 110 at closing time during the next year. All cash-flows occur at time 1.

Question 56. An option pays $|S_T - 90|$ at time T . Express its price in terms of the prices of call options and zero-coupon bonds. (You are not allowed to use put options!)

Question 57. A contract pays

$$S_T - 90 \text{ for } S_T < 100, S_T - 100 \text{ for } S_T \geq 100.$$

Express its price in terms of the prices of call options and zero-coupon bonds.

Question 58. You have access to the prices of zero-coupon bonds for all maturities, S_0 , the value of a stock today, and the prices, $C(K, T)$, of call options for all strikes K and maturities T . The stock is dividend-paying and has price process S_t . At time T , a contract pays

$$|S_T - 100|$$

if this number is less than 10 and zero otherwise. Find the price of the contract in terms of the data given. If insufficient data has been given, then explain what additional data is necessary and then derive the price.

Chapter 8

Question 59. Let S be a non-dividend paying stock. Order the prices of the following contracts as far as possible making only the assumption of no arbitrage.

1. A European call option on S struck at 100 with expiry 1.
2. A European call option on S struck at 101 with expiry 1.
3. A European call option on S struck at 101 with expiry 2.
4. An American call option on S struck at 100 with expiry 1.
5. A down-and-out call option with strike at 101 and barrier at 90 which is continuously monitored. One year expiry.
6. A down-and-out call option with strike at 101 and barrier at 90 which is monitored at times 0.25, 0.5 and 0.75. One year expiry.

Question 60. An asset price follows the process

$$\frac{dS_t}{S_t} = \frac{1}{2}\sigma^2 dt + \sigma dW_t,$$

with $S_0 = 100, T = 1$. Find an expression in terms of the cumulative normal for the probabilities of the following events:

1. $S_T > 100$;
2. $S_T < 81$;
3. $\min_{t \in [0, T]} S_t < 90$;
4. $\min_{t \in [0, T]} S_t < 90, S_T > 100$.

Question 61. A stock follows the process

$$dS_t = \sigma dW_t,$$

with $S_0 = 0$. Find an expression in terms of the cumulative normal for the probability both the following occur:

$$S_T > 0$$

and the minimum of S_t on $[0, T]$ is less than -1 .

Question 62. Let X_t be the exchange rate between the Australian dollar and the British pound. Let S_t be the price of a non-dividend paying Australian stock. For each of X_t and S_t order the prices of the following contracts as far as possible. All strikes are at the forward price.

1. A European put option with one-year maturity.
2. An American call option with one-year maturity.
3. A European put option with two-year maturity.
4. An American put option with two-year maturity.
5. An American call option with one-year maturity that knocks out if the stock falls below $2/3$ of its initial value.

Give brief justification.

Question 63. A contract is worth zero today. Its price obeys

$$dX_t = dW_t$$

with W_t a driftless Brownian motion. A riskless bond can be freely traded at all times and is always worth 1. A derivative contract D , pays 2 at time 2 if during $0 < t < 2$, X_t passes below -1 , then above 1 and is below 0 at time 2. It pays 1 otherwise. Develop a formula for the price of D .

Chapter 9

Question 64. A barrier option knocks-out only when the spot has been behind the barrier for 30 consecutive days. Discuss how to price this with Monte Carlo and trees in the Black–Scholes model.

Question 65. A barrier option knocks-out only when the spot has been behind the barrier for a total of 30 days. Discuss how to price this with Monte Carlo and trees in the Black–Scholes model. How will its price compare with the option that requires 30 consecutive days to knock out?

Question 66. Let S_t be the price process of a dividend-paying stock. For $t = t_1, t_2$, with $0 = t_0 < t_1 < t_2$, you can observe today (i.e., time 0) the price of a zero-coupon bond with expiry t , $Z(t)$, the implied volatility, $\hat{\sigma}_t$, of an at-the-money call option expiring at time t , and the forward price f_t for time t . Explain how you would implement a pricing model for the following contract.

- The investor pays 1 to the bank at time zero.
- At time t_1 , if $S_{t_1} < S_{t_0}$ the investor receives 1 and the deal terminates. Otherwise the investor receives a coupon C_1 and the deal continues.
- At time t_2 , the investor receives $1 + S_{t_2}/S_{t_0}$.

The model should return a positive number if the deal is profitable to the bank. Your model should give a price in terms of simple mathematical functions (possibly including the cumulative normal and the bivariate cumulative normal) as functions of the market-observable inputs.

Question 67. A derivative D pays a function f of the stock price at time T , where

$$f(S_T) = \begin{cases} 0 & \text{for } S_T < 100, \\ 1 & \text{for } 100 \leq S_T < 110, \\ S_T - 109 & \text{for } S_T \geq 110. \end{cases}$$

If we can observe the implied volatilities of options of all strikes, and the discount factor for time T , give the price of D in terms of the Black–Scholes formula, its Greeks, cumulative normals and the market-given information.

Question 68. Find upper triangular pseudo-square roots of the following matrices

$$\begin{pmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 6 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

Question 69. Find a pseudo-square root of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix}.$$

Question 70. A stock is non-dividend paying and has price process S_t . The stock price, S_0 , is 100. A riskless bond exists with continuously compounding rate 0.0. Call options struck at 100 with expiry 1 and 2 have implied volatilities 12 and 15 percent respectively. Find the price of a contract paying $(S_2/S_1 - 1)_+$ at time 2 in an appropriate model. Give your answer in terms of the cumulative normal function.

Question 71. The Black–Scholes model is assumed. A stock is dividend-paying with positive dividend rate d . Interest rates are positive. A knock-in down-and-in American barrier call option has expiry T , barrier H and strike K . Thus the option can be exercised at any time before T provided the stock has been below H at least once during the life of the contract.

- Discuss how you would construct an efficient numerical method to approximate the Black–Scholes price for this contract.
- How would the price of this contract compare to that of an American call option without the barrier feature?
- Would the pricing be easier if the dividend-rate were zero?

Question 72. It is your first day working at VillageBank. They have decided to trade exotic foreign exchange options. You are asked to develop a Monte Carlo pricer. Your predecessor has left you a random number generator that produces standard normal random numbers and not much else. They have subscribed to Bloomsbury and can get the following data

- the prices of ZCBs for all maturities for both currencies,
- the price of call options for all maturities struck at the forward.

Discuss how you would build and calibrate a model to price the following contracts

- an Asian option with weekly monitoring dates and two years expiry,
- a discrete barrier option with daily monitoring which is a down and out call, expiry is two years.

How would the accuracy vary with the number of paths used? How else could the accuracy be increased?

Chapter 10

Chapter 11

Question 73. The non-dividend paying stock prices X_t and Y_t follow geometric Brownian motions. A riskless bond growing at rate r exists. The following contracts all pay out at time T . We have $0 < s < T$. Find the price at time zero of contracts with the following pay-offs:

- $X_T Y_T I_{X_T > K}$;
- $X_T I_{Y_s > H}$.

Question 74. The non-dividend paying stock prices X_t and Y_t follow geometric Brownian motions. A riskless bond growing at rate r exists. Let $t_1 < t_2 < t_3$. Find the price at time zero of a contract that pays

$$X_{t_2} Y_{t_2} I_{X_{t_1} > K}$$

at time t_3 .

Question 75. A pair of non-dividend paying stocks, (X, Y) , is modelled by a two-dimensional Black–Scholes model with correlation coefficient ρ . A quant elects to discretize the correlated underlying Brownian motions by four points $(-\alpha, -\alpha)$, $(-\alpha, \alpha)$, $(\alpha, -\alpha)$ and (α, α) , taking probabilities p, q, q and p , respectively. Find values of α, p and q for a time-step of length Δt which will make the model converge correctly as Δt tends to zero.

Question 76. The non-dividend paying stock prices X_t and Y_t follow geometric (jointly normal) Brownian motions with drifts μ_X, μ_Y and volatilities σ_X , and σ_Y . The correlation coefficient is $\rho > 0$. A riskless bond growing at rate r exists. The following contracts all pay out at time T . Using the multi-dimensional Black–Scholes model, find the prices at time zero of contracts with the following pay-offs:

- X_T/Y_T ;
- $(X_T - Y_T)I_{Y_T > H}$.

Question 77. The USD dollar stock X follows geometric Brownian motion with volatility 0.1 and drift 0.1. Its value today is 100 USD. The value of an AUD in USD is 0.8, and follows geometric Brownian motion. Its drift is 0 and its volatility is 0.2. The two Brownian motions are independent. The AUD continuously compounding risk-free rate is 5% and the USD continuously compounding risk-free rate is 3%.

- Give the price of an option to buy X for 125 AUD with one year expiry in terms of the Black–Scholes formula.
- For what strike in AUD would a call and put have the same value?
- For that strike, give an approximate price in USD.

Question 78. Three non-dividend paying stocks' price processes $S_t^{(j)}$ obey the multi-dimensional Black-Scholes model. A contract D pays

$$\left(\frac{S_T^{(1)} S_T^{(2)}}{S_T^{(3)}} - S_T^{(3)} \right)_+$$

at time T . Find a formula for the rho of the price of D . (The rho is the derivative of the price with respect to the risk-free rate.)

Question 79. Two non-dividend paying stocks X_t and Y_t satisfy Black-Scholes assumptions with volatilities σ_X and σ_Y . However X_t is denominated in USD and Y_t is denominated in AUD. Let F_t denote the value of one US dollar in AUD at time t . Denote the continuously rate in AUD by r and that in USD by d . Take $d = 0$. Suppose F_t follows geometric Brownian motion with volatility σ_F . The Brownian motions are jointly normal with pairwise correlations ρ_{FX} , ρ_{FY} and ρ_{XY} . Develop a formula for an option that pays

$$(X_T - Y_T)_+$$

Australian dollars at time T .