

More Mathematical Finance

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Contents

Preface	xiii
Chapter 1. Optionality, convexity and volatility	1
1.1. Introduction	1
1.2. Volatility and convexity	1
1.3. Convexity and optionality	3
1.4. Is convexity necessary?	7
1.5. Key points	8
1.6. Further reading	8
1.7. Exercises	8
Chapter 2. Where does the money go?	9
2.1. Introduction	9
2.2. The money bleed	10
2.3. Analyzing the examples	14
2.4. Volatility convexity and the existence of smiles	17
2.5. Key Points	21
2.6. Further reading	21
2.7. Exercises	21
Chapter 3. The Bachelier model	23
3.1. Introduction	23
3.2. The pricing formula	23
3.3. Approximations and comparisons	26
3.4. Key points	27
3.5. Further reading	27
3.6. Exercises	27
Chapter 4. Deriving the Delta	29
4.1. Introduction	29
4.2. The stock measure	29

4.3. Homogeneity	30
4.4. Other cases	31
4.5. Key points	32
4.6. Further reading	32
4.7. Exercises	32
Chapter 5. Volatility derivatives and model-free dynamic replication	33
5.1. Introduction	33
5.2. Variance swaps	34
5.3. Pricing general volatility derivatives	36
5.4. Hedging a volatility derivative	38
5.5. Key points	40
5.6. Further reading	40
5.7. Exercises	40
Chapter 6. Credit derivatives	41
6.1. Introduction	41
6.2. The basic instruments	42
6.3. The philosophy of pricing credit derivatives	46
6.4. Hazard rates	48
6.5. Pricing simple credit instruments	50
6.6. Key points	50
6.7. Further reading	51
6.8. Exercises	51
Chapter 7. The Monte Carlo pricing of portfolio credit derivatives	53
7.1. Introduction	53
7.2. The Li model	54
7.3. Importance sampling for basket default swaps	56
7.4. Tranched CDOs by Monte Carlo	59
7.5. The default density in the Li Model	62
7.6. The likelihood ratio method for basket credit derivatives	63
7.7. The pathwise method for n th-to-default swaps	65
7.8. Key points	67
7.9. Further reading	67
7.10. Exercises	68
Chapter 8. Quasi-analytic methods for pricing portfolio credit derivatives	71
8.1. Introduction	71
8.2. The loss distribution for independent defaults	72

8.3. Computing the loss distribution in a single-factor model	73
8.4. Turning loss distributions into prices	74
8.5. Stochastic recovery rates	76
8.6. The Fourier transform approach	77
8.7. Bucketing	78
8.8. Key points	79
8.9. Further reading	80
8.10. Exercises	80
Chapter 9. Implied correlation for portfolio credit derivatives	81
9.1. Introduction	81
9.2. Implied correlations	82
9.3. Base correlation	85
9.4. Mapping methodologies	87
9.5. Hedging and the computation of Greeks	89
9.6. Key points	91
9.7. Further reading	92
9.8. Exercises	92
Chapter 10. Alternate models for portfolio credit derivatives	93
10.1. Introduction	93
10.2. Random factor loadings	95
10.3. Elliptic copulas	100
10.4. Multiple default processes	102
10.5. Intensity Gamma	104
10.6. Key points	111
10.7. Further reading	111
10.8. Exercises	111
Chapter 11. The non-commutativity of discretization	113
11.1. Introduction	113
11.2. Discretization and risk-neutrality	113
11.3. Discretization and Greeks	117
11.4. Factor reduction	120
11.5. Importance sampling	122
11.6. Coordinate changes	123
11.7. Calibration	124
11.8. Key points	127
11.9. Further reading	127
11.10. Exercises	127

Chapter 12. What is a factor?	129
12.1. Introduction	129
12.2. Factors for an implementation of the LMM	130
12.3. Factor reduction	132
12.4. The number of common factors	136
12.5. The dimension of the space attainable	138
12.6. Markovian dimension with drifts	142
12.7. Markov functional models	144
12.8. Matrix separability	146
12.9. Key points	148
12.10. Further reading	148
12.11. Exercises	149
Chapter 13. Early exercise and Monte Carlo Simulation	151
13.1. Introduction	151
13.2. A sketch of the least-squares method	152
13.3. The details of the least-squares algorithm	153
13.4. Carrying out the regression	155
13.5. Breaking a contract	157
13.6. Assessing and extending least-squares	159
13.7. Upper bounds and the seller's price	160
13.8. Recharacterising the optimal hedge	163
13.9. Upper bounds for breakable contracts	165
13.10. Never exercise sub-optimally	166
13.11. Multiplicative upper bounds	167
13.12. Key points	172
13.13. Further reading	172
13.14. Exercises	172
Chapter 14. The Brownian bridge	175
14.1. Introduction	175
14.2. Reducing to the driftless case	175
14.3. The law of the minimum for a Brownian bridge	177
14.4. The distribution at intervening times	178
14.5. Using the Brownian bridge for path generation	180
14.6. The geometric bridge	181
14.7. Key points	183
14.8. Further reading	183
14.9. Exercises	183

Chapter 15. Quasi Monte Carlo Simulation	185
15.1. Introduction	185
15.2. Choices and more choices	187
15.3. The proper use of Sobol numbers	192
15.4. Assessing convergence	200
15.5. Key points	205
15.6. Further reading	205
15.7. Exercises	205
Chapter 16. Pricing continuous barrier options using a jump-diffusion model	207
16.1. Introduction	207
16.2. The Merton jump-diffusion model	209
16.3. Importance sampling and stratification	210
16.4. The price conditional on no jumps occurring	211
16.5. The algorithm	212
16.6. Numerical results	213
16.7. Key points	217
16.8. Further reading	218
16.9. Exercises	218
Chapter 17. The Fourier–Laplace transform and option pricing	219
17.1. Introduction	219
17.2. Definitions and basic results	219
17.3. Working with the log forward	228
17.4. The Fourier transform in log-strike space	233
17.5. The time-value approach	239
17.6. The probability decomposition approach	241
17.7. Working with characteristic functions	242
17.8. Known characteristic functions	244
17.9. The Heston characteristic function	247
17.10. Numerical implementation	249
17.11. Key points	251
17.12. Further reading	251
17.13. Exercises	251
Chapter 18. The cos method	253
18.1. Introduction	253
18.2. Cosine series	253
18.3. Cosine series and characteristic functions	255

18.4.	European option pricing	256
18.5.	Homogeneous models and the cos method	259
18.6.	Bermudan options	260
18.7.	American options	263
18.8.	Key points	264
18.9.	Further reading	264
18.10.	Exercises	264
Chapter 19.	What are market models?	265
19.1.	Introduction	265
19.2.	The general set-up	266
19.3.	Drifts and martingales	267
19.4.	Calibration	268
19.5.	Products	272
19.6.	Key points	279
19.7.	Further reading	280
19.8.	Exercises	280
Chapter 20.	Discounting in market models	281
20.1.	Introduction	281
20.2.	Possible numeraires	282
20.3.	The most common choices and their consequences	284
20.4.	Using the numeraire to discount	286
20.5.	Numeraire matching, variance reduction and discretization bias	288
20.6.	Forward discounting in the spot measure	289
20.7.	Key points	290
20.8.	Further reading	291
20.9.	Exercises	291
Chapter 21.	Drifts again	293
21.1.	Introduction	293
21.2.	Rapid computation of drifts	293
21.3.	Evolving the bond	295
21.4.	Positivity issues with bond evolution	297
21.5.	Predictor corrector	299
21.6.	Stopping predictor corrector	299
21.7.	Pietersz-Pelsser-Regenmortel	301
21.8.	Numerical comparisons of drift methods	303
21.9.	Key points	305
21.10.	Further reading	306

21.11. Exercises	306
Chapter 22. Adjoint and automatic Greeks	307
22.1. Introduction	307
22.2. Model Deltas using the Giles–Glasserman method	308
22.3. Pathwise Vegas in the LMM using the Giles–Glasserman method	311
22.4. The adjoint acceleration	313
22.5. The LMM as a sequence of vector operations	320
22.6. The limitations of the adjoint method	322
22.7. Forwards versus backwards	323
22.8. Key points	324
22.9. Further reading	324
22.10. Exercises	324
Chapter 23. Estimating correlation for the LIBOR market model	327
23.1. Introduction	327
23.2. The set-up	327
23.3. Time parameterization	328
23.4. Interactions with boot-strapping	329
23.5. Factor reduction	331
23.6. Other market models	332
23.7. Time-series step size	332
23.8. Correlation smoothing	333
23.9. Does it really matter?	337
23.10. Key points	338
23.11. Further reading	338
23.12. Exercises	338
Chapter 24. Swap-rate market models	341
24.1. Introduction	341
24.2. Deducing the bond-ratios for the co-terminal model	342
24.3. Cross-variation derivative	343
24.4. Swap-rate drift computations	346
24.5. Constant maturity market models	348
24.6. Co-initial swap-rates	350
24.7. Incremental market models	352
24.8. Calibrating the co-terminal swap-rate market model	356
24.9. Evolving swap-rates	357
24.10. LIBOR versus swap-rate market models	358
24.11. Key points	359

24.12.	Further reading	360
24.13.	Exercises	360
Chapter 25.	Calibrating market models	363
25.1.	Introduction	363
25.2.	Understanding pseudo-square roots	365
25.3.	Decomposing pseudo-roots	367
25.4.	Time dependence and factor maintenance	368
25.5.	Mapping between models and swaption approximations	368
25.6.	Cascade calibration	371
25.7.	Fitting caplets and co-terminal swaptions	374
25.8.	Rescaling and LMM calibration	381
25.9.	Period mismatch	383
25.10.	Global optimization	385
25.11.	Calibration with displacements	386
25.12.	Key points	387
25.13.	Further reading	388
25.14.	Exercises	388
Chapter 26.	Cross-currency market models	389
26.1.	Introduction	389
26.2.	Notation	390
26.3.	Dynamics	390
26.4.	Understanding calibration	393
26.5.	Pricing given a calibration	395
26.6.	Approximation formulas for the volatility of the forward FX rate	396
26.7.	Equity-linked notes	397
26.8.	Key points	398
26.9.	Further reading	399
26.10.	Exercises	399
Chapter 27.	Mixture models	401
27.1.	Introduction	401
27.2.	Uncertain parameter models	402
27.3.	As a smoothing methodology	403
27.4.	The advantages and disadvantages	403
27.5.	Key points	404
27.6.	Further reading	405
27.7.	Exercises	405

Chapter 28. The convergence of binomial trees	407
28.1. Introduction	407
28.2. Richardson extrapolation	408
28.3. Convergence of simple trees for European options	412
28.4. Convergence theorems	414
28.5. Redesigning trees	415
28.6. The Leisen–Reimer tree	417
28.7. Higher order convergence	419
28.8. Code for higher order trees	420
28.9. More and more trees	422
28.10. Choices for trees	425
28.11. American options	426
28.12. Assessing accuracy	428
28.13. Truncation choices	429
28.14. Key points	430
28.15. Further reading	430
28.16. Exercises	430
Chapter 29. Asymmetry in option pricing	433
29.1. Introduction	433
29.2. American optionality	434
29.3. Incomplete markets	437
29.4. Transaction costs	439
29.5. Key points	441
29.6. Further reading	441
29.7. Exercises	441
Chapter 30. A perfect model?	443
30.1. Introduction	443
30.2. The vanilla options trader	444
30.3. Dynamic hedging with a perfect model	445
30.4. The portfolio	446
30.5. The exotics trader	447
30.6. Key points	447
30.7. Further reading	448
30.8. Exercises	448
Chapter 31. The fundamental theorem of asset pricing.	449
31.1. Introduction	449
31.2. The easy direction	450

31.3.	The hard direction in the discrete case	451
31.4.	Attaining the minimal price	454
31.5.	Key points	456
31.6.	Further reading	456
31.7.	Exercises	456
Appendix A.	The discrete Fourier transform	457
A.1.	Introduction	457
A.2.	Roots of unity	457
A.3.	The discrete Fourier transform	460
A.4.	The fast Fourier transform	462
A.5.	The discrete Fourier transform and convolutions	463
A.6.	The fast Fourier transform and matrix multiplication	464
A.7.	Key points	466
A.8.	Further reading	466
Bibliography		467
Index		477

Preface

It is now ten years since the first draft of “the Concepts and Practice of Mathematical Finance” was finished. The volume of research published during that time has been immense. New areas have arisen and many questions have been resolved. Some markets such as portfolio credit derivatives have arisen, boomed and crashed. “More Mathematical Finance” is therefore a sequel, and it is intended to be a second or third book on financial mathematics. In particular, rather than recall basic theory, I will refer to “Concepts” as much as possible in order to minimize overlap and maximize the amount of new material.

This sequel is not intended to be comprehensive. The field is now far too large for such an undertaking to be practical. In any case, I am a firm believer in “write what you know.” Most of the topics in the book are related to my own research in one way or another, and I hope to pass on some of the insights I have gained from using and implementing these models. To me that is the essence of the book, my objective is to give the reader my own personal perspectives on how one should view various issues. Thus whilst most of the mathematics and models here presented can be found somewhere in the literature, the perspectives I present often cannot.

Much of the book focuses on numerical methods. A pricing model is not much use unless it can be implemented and calibrated. The ability to compute Greeks is another essential. My objective is therefore to show how the mathematics can be translated into an implementable, usable model. However, this is not a recipe book. Although I present algorithms, my objective is to give the reader an understanding that makes the algorithms clear, rather than to present a piece of pseudo-code to copy out. I will, however, occasionally point to where the relevant code can be found in the QuantLib open source library. I largely avoid presenting purely numerical techniques which are well known outside finance. For example, I leave the details of how to carry out Gaussian integration to other texts. However, I do present extensive discussion of how to use Sobol numbers for quasi-Monte Carlo simulation, since this seems to be a much misunderstood topic.

I have restricted this text to mathematical finance in the sense of derivatives pricing. A more specific but rather unwieldy title might have been “how to think about some numerical techniques for pricing derivative contracts.”

At some point, one must call a halt to writing, and many topics that were considered have not made it in to this book. These include Levy processes, OIS discounting, SABR, asymptotic expansion approximations, solving SDEs, numerical methods for solving PDEs, short rate models, the HJM model, commodities, power derivatives, CGMYSV, GPUs, proxy methods for Greek computation, interpolation methodologies for interest rates, local volatility, firm-value models, VAR, CES, mean-variance theory, CAPM, utility theory, APT ... The list is endless. Eventually, when I again feel that I have enough to say to justify another book, it will be time to write “Even More Mathematical Finance.”

So what topics are covered? I spend four chapters on portfolio credit derivatives since it is an area that has gone from obscurity to fame to notoriety in the last few years. I look at binomial trees in depth since it is a topic which is much misunderstood: we will see that there are at least twenty different ways to place the nodes of a tree, and that each of these can be implemented in at least sixteen different ways. Monte Carlo techniques are examined in depth with chapters on the Brownian bridge, quasi-Monte Carlo, the early exercise problem and stratification.

Market models for pricing exotic interest rate derivatives have been my principal research interest for many years. This is reflected by chapters on their applicability, drift computation and approximation, correlation estimation, swap-rate market models, calibration, discounting and cross-currency market models. The chapters on discretization, factor reduction, quasi-Monte-Carlo and computing sensitivities with adjoint methods whilst written in a more general context are also directly relevant to market models.

I also include a few chapters on more philosophical questions. These include chapters on asymmetry, evaluating a perfect model, the fundamental theorem of asset pricing, convexity, mixture models and the money bleed.

Certain chapters have been included simply because I think the results and/or the mathematics are neat and I want them to share them with the reader. These include a chapter on how to differentiate the Black–Scholes formula, one on volatility derivatives and the Bachelier model. Some chapters which are both neat and very numerical are on the Fourier transform, the cos method and importance sampling with jump-diffusion models.

Few readers have the time and inclination to read a long book from start to finish and I have therefore tried to make individual chapters as independent as

possible. However, there are inevitably some dependencies and I now discuss these. First, the credit chapters should be read in order. Second, the introduction to market models should be read before the other market models chapters; these are, however, largely independent of each other. The cross-currency market model chapter (26) does, of course, assume familiarity with market models. Third, the cos chapter (18) relies on the Fourier transform chapter (17). The quasi-Monte Carlo chapter (15) and the importance sampling chapter (16) both depend on the Brownian bridge chapter (14). The remaining chapters are largely stand alone and can be read in any order.

The website for this book is

www.markjoshi.com/more

visit there for updates, questions, new editions, typos and news. Please use the forum there to ask questions about the text and to inform me of typos.

I have included end of chapter exercises. These take a variety of forms ranging from simple computations to complicated proofs. Many of them are more computer projects than exercises since ultimately this book is about modelling. I have not included solutions, but you are encouraged to discuss the problems on the book's website.

Various versions of the manuscript have been read by a rather large number of people and I thank them all for their comments. The readers include Barbara La Scala, Will Wright, Chris Beveridge, Jiun Hong Chan, Stephen Chin, Nick Denson, Andrew Downes, Robert Tang, Chao Yang, Ferdinando Ametrano, Paulius Jakubenas, Harry Lo, Lew Burton, Agustin Lebron, Oh Kang Kwon, Alan Lewis, Nagulan Saravanamuttu, Graeme West, Dherminder Kainth and Lorenzo Liesch, as well as many others.

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