More Mathematical Finance

Mark S. Joshi

University of Melbourne

PILOT WHALE PRESS
Contents

Preface xiii

Chapter 1. Optionality, convexity and volatility 1
  1.1. Introduction 1
  1.2. Volatility and convexity 1
  1.3. Convexity and optionality 3
  1.4. Is convexity necessary? 7
  1.5. Key points 8
  1.6. Further reading 8
  1.7. Exercises 8

Chapter 2. Where does the money go? 9
  2.1. Introduction 9
  2.2. The money bleed 10
  2.3. Analyzing the examples 14
  2.4. Volatility convexity and the existence of smiles 17
  2.5. Key Points 21
  2.6. Further reading 21
  2.7. Exercises 21

Chapter 3. The Bachelier model 23
  3.1. Introduction 23
  3.2. The pricing formula 23
  3.3. Approximations and comparisons 26
  3.4. Key points 27
  3.5. Further reading 27
  3.6. Exercises 27

Chapter 4. Deriving the Delta 29
  4.1. Introduction 29
  4.2. The stock measure 29
# CONTENTS

Chapter 12.  What is a factor?  
  12.1.  Introduction  
  12.2.  Factors for an implementation of the LMM  
  12.3.  Factor reduction  
  12.4.  The number of common factors  
  12.5.  The dimension of the space attainable  
  12.6.  Markovian dimension with drifts  
  12.7.  Markov functional models  
  12.8.  Matrix separability  
  12.9.  Key points  
  12.10.  Further reading  
  12.11.  Exercises  

Chapter 13.  Early exercise and Monte Carlo Simulation  
  13.1.  Introduction  
  13.2.  A sketch of the least-squares method  
  13.3.  The details of the least-squares algorithm  
  13.4.  Carrying out the regression  
  13.5.  Breaking a contract  
  13.6.  Assessing and extending least-squares  
  13.7.  Upper bounds and the seller’s price  
  13.8.  Recharacterising the optimal hedge  
  13.9.  Upper bounds for breakable contracts  
  13.10.  Never exercise sub-optimally  
  13.11.  Multiplicative upper bounds  
  13.12.  Key points  
  13.13.  Further reading  
  13.14.  Exercises  

Chapter 14.  The Brownian bridge  
  14.1.  Introduction  
  14.2.  Reducing to the driftless case  
  14.3.  The law of the minimum for a Brownian bridge  
  14.4.  The distribution at intervening times  
  14.5.  Using the Brownian bridge for path generation  
  14.6.  The geometric bridge  
  14.7.  Key points  
  14.8.  Further reading  
  14.9.  Exercises  

Page numbers:
## CONTENTS

<table>
<thead>
<tr>
<th>Chapter 15. Quasi Monte Carlo Simulation</th>
<th>185</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.1. Introduction</td>
<td>185</td>
</tr>
<tr>
<td>15.2. Choices and more choices</td>
<td>187</td>
</tr>
<tr>
<td>15.3. The proper use of Sobol numbers</td>
<td>192</td>
</tr>
<tr>
<td>15.4. Assessing convergence</td>
<td>200</td>
</tr>
<tr>
<td>15.5. Key points</td>
<td>205</td>
</tr>
<tr>
<td>15.6. Further reading</td>
<td>205</td>
</tr>
<tr>
<td>15.7. Exercises</td>
<td>205</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 16. Pricing continuous barrier options using a jump-diffusion model</th>
<th>207</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.1. Introduction</td>
<td>207</td>
</tr>
<tr>
<td>16.2. The Merton jump-diffusion model</td>
<td>209</td>
</tr>
<tr>
<td>16.3. Importance sampling and stratification</td>
<td>210</td>
</tr>
<tr>
<td>16.4. The price conditional on no jumps occurring</td>
<td>211</td>
</tr>
<tr>
<td>16.5. The algorithm</td>
<td>212</td>
</tr>
<tr>
<td>16.6. Numerical results</td>
<td>213</td>
</tr>
<tr>
<td>16.7. Key points</td>
<td>217</td>
</tr>
<tr>
<td>16.8. Further reading</td>
<td>218</td>
</tr>
<tr>
<td>16.9. Exercises</td>
<td>218</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 17. The Fourier–Laplace transform and option pricing</th>
<th>219</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.1. Introduction</td>
<td>219</td>
</tr>
<tr>
<td>17.2. Definitions and basic results</td>
<td>219</td>
</tr>
<tr>
<td>17.3. Working with the log forward</td>
<td>228</td>
</tr>
<tr>
<td>17.4. The Fourier transform in log-strike space</td>
<td>233</td>
</tr>
<tr>
<td>17.5. The time-value approach</td>
<td>239</td>
</tr>
<tr>
<td>17.6. The probability decomposition approach</td>
<td>241</td>
</tr>
<tr>
<td>17.7. Working with characteristic functions</td>
<td>242</td>
</tr>
<tr>
<td>17.8. Known characteristic functions</td>
<td>244</td>
</tr>
<tr>
<td>17.9. The Heston characteristic function</td>
<td>247</td>
</tr>
<tr>
<td>17.10. Numerical implementation</td>
<td>249</td>
</tr>
<tr>
<td>17.11. Key points</td>
<td>251</td>
</tr>
<tr>
<td>17.12. Further reading</td>
<td>251</td>
</tr>
<tr>
<td>17.13. Exercises</td>
<td>251</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 18. The cos method</th>
<th>253</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.1. Introduction</td>
<td>253</td>
</tr>
<tr>
<td>18.2. Cosine series</td>
<td>253</td>
</tr>
<tr>
<td>18.3. Cosine series and characteristic functions</td>
<td>255</td>
</tr>
<tr>
<td>Chapter</td>
<td>Section</td>
</tr>
<tr>
<td>------------------</td>
<td>--------------------------------------------------------------------------</td>
</tr>
<tr>
<td>18.4</td>
<td>European option pricing</td>
</tr>
<tr>
<td>18.5</td>
<td>Homogeneous models and the cos method</td>
</tr>
<tr>
<td>18.6</td>
<td>Bermudan options</td>
</tr>
<tr>
<td>18.7</td>
<td>American options</td>
</tr>
<tr>
<td>18.8</td>
<td>Key points</td>
</tr>
<tr>
<td>18.9</td>
<td>Further reading</td>
</tr>
<tr>
<td>18.10</td>
<td>Exercises</td>
</tr>
<tr>
<td>19.1</td>
<td>What are market models?</td>
</tr>
<tr>
<td>19.2</td>
<td>Introduction</td>
</tr>
<tr>
<td>19.3</td>
<td>The general set-up</td>
</tr>
<tr>
<td>19.4</td>
<td>Drifts and martingales</td>
</tr>
<tr>
<td>19.5</td>
<td>Calibration</td>
</tr>
<tr>
<td>19.6</td>
<td>Products</td>
</tr>
<tr>
<td>19.7</td>
<td>Key points</td>
</tr>
<tr>
<td>19.8</td>
<td>Further reading</td>
</tr>
<tr>
<td>19.9</td>
<td>Exercises</td>
</tr>
<tr>
<td>20.1</td>
<td>Discounting in market models</td>
</tr>
<tr>
<td>20.2</td>
<td>Introduction</td>
</tr>
<tr>
<td>20.3</td>
<td>Possible numeraires</td>
</tr>
<tr>
<td>20.4</td>
<td>The most common choices and their consequences</td>
</tr>
<tr>
<td>20.5</td>
<td>Using the numeraire to discount</td>
</tr>
<tr>
<td>20.6</td>
<td>Numeraire matching, variance reduction and discretization bias</td>
</tr>
<tr>
<td>20.7</td>
<td>Forward discounting in the spot measure</td>
</tr>
<tr>
<td>20.8</td>
<td>Key points</td>
</tr>
<tr>
<td>20.9</td>
<td>Further reading</td>
</tr>
<tr>
<td>21.1</td>
<td>Drifts again</td>
</tr>
<tr>
<td>21.2</td>
<td>Rapid computation of drifts</td>
</tr>
<tr>
<td>21.3</td>
<td>Evolving the bond</td>
</tr>
<tr>
<td>21.4</td>
<td>Positivity issues with bond evolution</td>
</tr>
<tr>
<td>21.5</td>
<td>Predictor corrector</td>
</tr>
<tr>
<td>21.6</td>
<td>Stopping predictor corrector</td>
</tr>
<tr>
<td>21.7</td>
<td>Pietersz-Pelsser-Regenmortel</td>
</tr>
<tr>
<td>21.8</td>
<td>Numerical comparisons of drift methods</td>
</tr>
<tr>
<td>21.9</td>
<td>Key points</td>
</tr>
<tr>
<td>21.10</td>
<td>Further reading</td>
</tr>
</tbody>
</table>
21.11. Exercises

Chapter 22. Adjoint and automatic Greeks

22.1. Introduction
22.2. Model Deltas using the Giles–Glasserman method
22.3. Pathwise Vegas in the LMM using the Giles–Glasserman method
22.4. The adjoint acceleration
22.5. The LMM as a sequence of vector operations
22.6. The limitations of the adjoint method
22.7. Forwards versus backwards
22.8. Key points
22.9. Further reading
22.10. Exercises

Chapter 23. Estimating correlation for the LIBOR market model

23.1. Introduction
23.2. The set-up
23.3. Time parameterization
23.4. Interactions with boot-strapping
23.5. Factor reduction
23.6. Other market models
23.7. Time-series step size
23.8. Correlation smoothing
23.9. Does it really matter?
23.10. Key points
23.11. Further reading
23.12. Exercises

Chapter 24. Swap-rate market models

24.1. Introduction
24.2. Deducing the bond-ratios for the co-terminal model
24.3. Cross-variation derivative
24.4. Swap-rate drift computations
24.5. Constant maturity market models
24.6. Co-initial swap-rates
24.7. Incremental market models
24.8. Calibrating the co-terminal swap-rate market model
24.9. Evolving swap-rates
24.10. LIBOR versus swap-rate market models
24.11. Key points
Chapter 25. Calibrating market models

25.1. Introduction
25.2. Understanding pseudo-square roots
25.3. Decomposing pseudo-roots
25.4. Time dependence and factor maintenance
25.5. Mapping between models and swaption approximations
25.6. Cascade calibration
25.7. Fitting caplets and co-terminal swaptions
25.8. Rescaling and LMM calibration
25.9. Period mismatch
25.10. Global optimization
25.11. Calibration with displacements
25.12. Key points
25.13. Further reading
25.14. Exercises

Chapter 26. Cross-currency market models

26.1. Introduction
26.2. Notation
26.3. Dynamics
26.4. Understanding calibration
26.5. Pricing given a calibration
26.6. Approximation formulas for the volatility of the forward FX rate
26.7. Equity-linked notes
26.8. Key points
26.9. Further reading
26.10. Exercises

Chapter 27. Mixture models

27.1. Introduction
27.2. Uncertain parameter models
27.3. As a smoothing methodology
27.4. The advantages and disadvantages
27.5. Key points
27.6. Further reading
27.7. Exercises
Chapter 28. The convergence of binomial trees 407
  28.1. Introduction 407
  28.2. Richardson extrapolation 408
  28.3. Convergence of simple trees for European options 412
  28.4. Convergence theorems 414
  28.5. Redesigning trees 415
  28.6. The Leisen–Reimer tree 417
  28.7. Higher order convergence 419
  28.8. Code for higher order trees 420
  28.9. More and more trees 422
  28.10. Choices for trees 425
  28.11. American options 426
  28.12. Assessing accuracy 428
  28.13. Truncation choices 429
  28.14. Key points 430
  28.15. Further reading 430
  28.16. Exercises 430

Chapter 29. Asymmetry in option pricing 433
  29.1. Introduction 433
  29.2. American optionality 434
  29.3. Incomplete markets 437
  29.4. Transaction costs 439
  29.5. Key points 441
  29.6. Further reading 441
  29.7. Exercises 441

Chapter 30. A perfect model? 443
  30.1. Introduction 443
  30.2. The vanilla options trader 444
  30.3. Dynamic hedging with a perfect model 445
  30.4. The portfolio 446
  30.5. The exotics trader 447
  30.6. Key points 447
  30.7. Further reading 448
  30.8. Exercises 448

Chapter 31. The fundamental theorem of asset pricing. 449
  31.1. Introduction 449
  31.2. The easy direction 450
31.3. The hard direction in the discrete case 451
31.4. Attaining the minimal price 454
31.5. Key points 456
31.6. Further reading 456
31.7. Exercises 456

Appendix A. The discrete Fourier transform 457
A.1. Introduction 457
A.2. Roots of unity 457
A.3. The discrete Fourier transform 460
A.4. The fast Fourier transform 462
A.5. The discrete Fourier transform and convolutions 463
A.6. The fast Fourier transform and matrix multiplication 464
A.7. Key points 466
A.8. Further reading 466

Bibliography 467

Index 477
Preface

It is now ten years since the first draft of “the Concepts and Practice of Mathematical Finance” was finished. The volume of research published during that time has been immense. New areas have arisen and many questions have been resolved. Some markets such as portfolio credit derivatives have arisen, boomed and crashed. “More Mathematical Finance” is therefore a sequel, and it is intended to be a second or third book on financial mathematics. In particular, rather than recall basic theory, I will refer to “Concepts” as much as possible in order to minimize overlap and maximize the amount of new material.

This sequel is not intended to be comprehensive. The field is now far too large for such an undertaking to be practical. In any case, I am a firm believer in “write what you know.” Most of the topics in the book are related to my own research in one way or another, and I hope to pass on some of the insights I have gained from using and implementing these models. To me that is the essence of the book, my objective is to give the reader my own personal perspectives on how one should view various issues. Thus whilst most of the mathematics and models here presented can be found somewhere in the literature, the perspectives I present often cannot.

Much of the book focuses on numerical methods. A pricing model is not much use unless it can be implemented and calibrated. The ability to compute Greeks is another essential. My objective is therefore to show how the mathematics can be translated into an implementable, usable model. However, this is not a recipe book. Although I present algorithms, my objective is to give the reader an understanding that makes the algorithms clear, rather than to present a piece of pseudo-code to copy out. I will, however, occasionally point to where the relevant code can be found in the QuantLib open source library. I largely avoid presenting purely numerical techniques which are well known outside finance. For example, I leave the details of how to carry out Gaussian integration to other texts. However, I do present extensive discussion of how to use Sobol numbers for quasi-Monte Carlo simulation, since this seems to be a much misunderstood topic.
I have restricted this text to mathematical finance in the sense of derivatives pricing. A more specific but rather unwieldy title might have been “how to think about some numerical techniques for pricing derivative contracts.”

At some point, one must call a halt to writing, and many topics that were considered have not made it in to this book. These include Levy processes, OIS discounting, SABR, asymptotic expansion approximations, solving SDEs, numerical methods for solving PDEs, short rate models, the HJM model, commodities, power derivatives, CGMYSV, GPUs, proxy methods for Greek computation, interpolation methodologies for interest rates, local volatility, firm-value models, VAR, CES, mean-variance theory, CAPM, utility theory, APT ... The list is endless. Eventually, when I again feel that I have enough to say to justify another book, it will be time to write “Even More Mathematical Finance.”

So what topics are covered? I spend four chapters on portfolio credit derivatives since it is an area that has gone from obscurity to fame to notoriety in the last few years. I look at binomial trees in depth since it is a topic which is much misunderstood: we will see that there are at least twenty different ways to place the nodes of a tree, and that each of these can be implemented in at least sixteen different ways. Monte Carlo techniques are examined in depth with chapters on the Brownian bridge, quasi-Monte Carlo, the early exercise problem and stratification.

Market models for pricing exotic interest rate derivatives have been my principal research interest for many years. This is reflected by chapters on their applicability, drift computation and approximation, correlation estimation, swap-rate market models, calibration, discounting and cross-currency market models. The chapters on discretization, factor reduction, quasi-Monte-Carlo and computing sensitivities with adjoint methods whilst written in a more general context are also directly relevant to market models.

I also include a few chapters on more philosophical questions. These include chapters on asymmetry, evaluating a perfect model, the fundamental theorem of asset pricing, convexity, mixture models and the money bleed.

Certain chapters have been included simply because I think the results and/or the mathematics are neat and I want them to share them with the reader. These include a chapter on how to differentiate the Black–Scholes formula, one on volatility derivatives and the Bachelier model. Some chapters which are both neat and very numerical are on the Fourier transform, the cos method and importance sampling with jump-diffusion models.

Few readers have the time and inclination to read a long book from start to finish and I have therefore tried to make individual chapters as independent as
possible. However, there are inevitably some dependencies and I now discuss these. First, the credit chapters should be read in order. Second, the introduction to market models should be read before the other market models chapters; these are, however, largely independent of each other. The cross-currency market model chapter (26) does, of course, assume familiarity with market models. Third, the cos chapter (18) relies on the Fourier transform chapter (17). The quasi-Monte Carlo chapter (15) and the importance sampling chapter (16) both depend on the Brownian bridge chapter (14). The remaining chapters are largely stand alone and can be read in any order.

The website for this book is

www.markjoshi.com/more

visit there for updates, questions, new editions, typos and news. Please use the forum there to ask questions about the text and to inform me of typos.

I have included end of chapter exercises. These take a variety of forms ranging from simple computations to complicated proofs. Many of them are more computer projects than exercises since ultimately this book is about modelling. I have not included solutions, but you are encouraged to discuss the problems on the book’s website.

Various versions of the manuscript have been read by a rather large number of people and I thank them all for their comments. The readers include Barbara La Scala, Will Wright, Chris Beveridge, Jiun Hong Chan, Stephen Chin, Nick Dennis, Andrew Downes, Robert Tang, Chao Yang, Ferdinando Ametrano, Paulius Jakubenas, Harry Lo, Lew Burton, Agustin Lebron, Oh Kang Kwon, Alan Lewis, Nagulan Saravanamuttu, Graeme West, Dherminder Kainth and Lorenzo Liesch, as well as many others.

Mark Joshi
Melbourne, August 2011